Statistics 581, Problem Set 1

Wellner; 10/02/2002

Reading: Lehmann & Casella, TPE, pages 1 - 32; skim Chapter 0 handout; read Chapter 1 handout.
Due: Wednesday, October 9, 2002.

1. Let X and Y be i.i.d. Uniform$(0, 1)$ random variables Define $U = X - Y$, $V = \max(X, Y) = X \vee Y$.

   (i) What is the range of (U, V)?

   (ii) Find the joint density function $f_{U,V}(u, v)$ of the pair (U, V). Are U and V independent?

2. Ferguson, ACILST, #6, page 7. (This is know as the Polya-Cantelli lemma; see Chapter 2, Propostion 2.11, page 10.)

3. Suppose that for $\theta \in R$,

$$f_\theta(u, v) = \{1 + \theta(1 - 2u)(1 - 2v)\}1_{[0,1]^2}(u, v).$$

   (a) For what values of θ is f_θ a density function on $[0, 1]^2$?
   (b) For the set of θ's you identified in (a), find the corresponding distributon function F_θ and show that it has Uniform$(0, 1)$ marginal distributions.
   (c) If $(U, V) \sim F_\theta$, compute the correlation $\rho(U, V) \equiv \rho$ as a function of θ. Does this show any difficulty with this family of distributions as a model of dependence?

4. Suppose that F is the distribution function of random variables X, Y with $X \sim$ Uniform$(0, 1)$ marginally and $Y \sim$ Uniform$(0, 1)$ marginally. Thus $F(x, y) = P(X \le x, Y \le y)$ satisfies

$$F(x, 1) = x, \quad 0 \le x \le 1, \qquad \text{and} \qquad F(1, y) = y, \quad 0 \le y \le 1.$$

   (a) Show that

$$F(x, y) \le x \wedge y \equiv F_U(x, y)$$

for all $0 \le x \le 1$, $0 \le y \le 1$. Here $x \wedge y \equiv x$ if $x \le y$, y if $y \le x$.
(b) Show that

$$F(x,y) \ge (x+y-1)^+ \equiv F_L(x,y)$$

for all $0 \le x \le 1$, $0 \le y \le 1$. Here $z^+ = z 1_{[0,\infty)}(z)$.
(c) Show that F_U is the distribution function of (X,X) where $X \sim$ Uniform$(0,1)$. Show that F_L is the distribution function of $(X, 1-X)$ where $X \sim$ Uniform$(0,1)$.
(d) The distribution functions F_U and F_L are called the Fréchet bounds. Show that F_L and F_U are singular with respect to Lebesgue measure λ_2 on $[0,1]^2$; i.e. show that the corresponding probability measures P_L and P_U satisfy

$$P((X,Y) \in A) = 1, \qquad \lambda_2(A) = 0$$

and

$$P((X,Y) \in A^c) = 0, \qquad \lambda_2(A^c) = 1$$

for some set A (which will be different for P_L and P_U). This implies that F_L and F_U do not have densities with respect to Lebesgue measure on $[0,1]^2$. (See Chapter 0, Section 3, especially Definition 3.1 and Theorem 3.1.)

5. Lehmann and Casella, TPE, problem 1.10, page 62.