

Statistics 581, Problem Set 7 Solutions

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1. Consider the two parameter location-scale model

$$\mathcal{P} = \left\{ P_\theta : \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta \right\}$$

where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right),$$

and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ .

(a) Calculate the information matrix $I(\theta)$ for θ .

(b) For which of the densities in (a)-(e) of problem set #6, problem 3, is $I_{12}(\theta)$ not zero?

Solution: (a) This follows Lehmann and Casella, Example 6.5, pages 126-127. The score for location is

$$\begin{aligned} \dot{\mathbf{i}}_1(x) &= \frac{\partial}{\partial \theta_1} \log \left\{ \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) \right\} \\ &= -\frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \frac{1}{\theta_2} \end{aligned}$$

and the score for scale is

$$\begin{aligned} \dot{\mathbf{i}}_2(x) &= \frac{\partial}{\partial \theta_2} \log \left\{ \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) \right\} \\ &= -\frac{1}{\theta_2} - \frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \frac{(x - \theta_1)}{\theta_2^2} \end{aligned}$$

Thus we compute

$$\begin{aligned} I_{11}(\theta) &= E\dot{\mathbf{i}}_1^2(X) = \frac{1}{\theta_2^2} \int \left(\frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \right)^2 \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) dx \\ &= \frac{1}{\theta_2^2} \int \left(\frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{f,loc}, \\ I_{22}(\theta) &= E\dot{\mathbf{i}}_2^2(X) = \frac{1}{\theta_2^2} \int \left(-1 - \frac{(x - \theta_1)}{\theta_2} \frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \right)^2 \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) dx \\ &= \frac{1}{\theta_2^2} \int \left(-1 - y \frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{f,scal}, \\ I_{12}(\theta) &= E\dot{\mathbf{i}}_1(X)\dot{\mathbf{i}}_2(X) = I_{21}(\theta) \\ &= \frac{1}{\theta_2^2} \int \left(-\frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \right) \left(-1 - \frac{(x - \theta_1)}{\theta_2} \frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \right) \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) dx \\ &= \frac{1}{\theta_2^2} \int y \left(\frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{12,f}. \end{aligned}$$

(b) Note that $I_{12,f} = 0$ for all symmetric densities f since the score for location, $\dot{\mathbf{I}}_1(y) = -f'(y)/f(y)$ is odd $\dot{\mathbf{I}}_1(-y) = -\dot{\mathbf{I}}_1(y)$, while the score for scale, $\dot{\mathbf{I}}_2(y) = -1 - yf'(y)/f(y)$, is even ($\dot{\mathbf{I}}_2(-y) = \dot{\mathbf{I}}_2(y)$), and f is even. Thus $I_{12}(\theta) = 0$ for cases (a)-(d) in problem 3, problem set #6, while $I_{12}(\theta) = \theta_2^{-2}I_{12,f} \neq 0$ in case (e). I calculate

$$\begin{aligned} I_{12,f} &= \int y \left(\frac{f'}{f}(y) \right)^2 f(y) dy \\ &= \int_{-\infty}^{\infty} y(-1 + e^{-y})^2 e^{-y} \exp(-e^{-y}) dy \\ &= - \int_0^{\infty} (\log v)(v-1)^2 e^{-v} dv = -(1-\gamma) \end{aligned}$$

where γ is Euler's constant. In fact this is related to I_{12} that we calculated already for the Weibull family in Example 3.2.5 in the notes and in the first display at the top of page 2 of the Handout on Gamma, Digama, and Polygamma. Note that from problem 4, problem set #6, we also have

$$\begin{array}{ll} I_{f,loc} = 1 & \text{for } f = \phi, \text{ the } N(0,1) \text{ density,} \\ I_{f,loc} = 1/3 & \text{for } f = \text{the logistic density,} \\ I_{f,loc} = 1 & \text{for } f = \text{the Laplace density,} \\ I_{f,loc} = (k+1)/(k+2) & \text{for } f = \text{the } t_k \text{ density,} \\ I_{f,loc} = 1 & \text{for } f = \text{the Gumbel or extreme value density.} \end{array}$$

On the other hand, we find that $I_{f,scale}$ is given the the various cases as follows: (a) For the normal density $f = \phi$, $I_{f,scale} = \int (x^2 - 1)\phi(x)dx = Var(Z^2) = 2$ where $Z \sim N(0, 1)$.

(b) For the logistic density f , the information for scale is

$$\begin{aligned} I_{f,scale} &= \int_{-\infty}^{\infty} \left\{ x \left(\frac{1 - e^{-x}}{1 + e^{-x}} \right) - 1 \right\}^2 dF(x) \\ &= \int_{-\infty}^{\infty} \{x(2F(x) - 1) - 1\}^2 dF(x) = (3 + \pi^2)/9 \end{aligned}$$

as is given in detail by Johnson and Kotz (1970) and by deCani and Stine (1986).

(c) For the double exponential (or Laplace) density $-(f'/f)(x) = \text{sign}(x)$, so $I_{f,scale} = 1$.

(d) For the t_k density, by using a change of variables much as in the location case,

$$I_{f,scale} = \frac{2k}{k+3};$$

note that $I_{t_k,scale} = (2k)/(k+3) \rightarrow 2 = I_{N(0,1),scale}$ as $k \rightarrow \infty$.

(e) For the extreme value distribution, $F(x) = \exp(-\exp(-x))$, and therefore if $X \sim F$ the random variable $Y = \exp(-X) \sim \text{exponential}(1)$:

$$P(Y \geq y) = P(\exp(-X) \geq y) = P(X \leq -\log(y)) = \exp(-\exp(\log y)) = \exp(-y).$$

Since $-(f'/f)(x) = -1 + e^{-x}$, it is easily seen that

$$\begin{aligned} I_f &= E\{-1 - X(f'/f)(X)\}^2 = E\{(X(1 - e^{-X}) - 1)\}^2 \\ &= E\{(\log Y)(Y - 1) - 1\}^2 = \frac{\pi^2}{6} + (1 - \gamma)^2 \end{aligned}$$

as in our information calculations for the Weibull family.

2. Lehmann and Casella, TPE, Problem 6.6, page 142. If P_θ is the probability measure with density p_θ with respect to Lebesgue measure on \mathbb{R} given by

$$p_\theta(x) = (1 - \epsilon)\phi(x - \xi) + \frac{\epsilon}{\tau}\phi\left(\frac{x - \xi}{\tau}\right)$$

where ϕ is the standard normal density and $\theta = (\xi, \tau, \epsilon) \in \mathbb{R} \times \mathbb{R}^+ \times [0, 1]$, find the information matrix $I(\theta) = I(\xi, \tau, \epsilon)$.

Solution: I would prefer to write $\theta = (\xi, \epsilon, \tau)$. The first thing to note about the model

$$\mathcal{P} = \{P_\theta : (dP_\theta/d\lambda)(x) = p(x; \theta) = p(x)\}$$

is that it is a location family: $p(x; \xi, \epsilon, \tau) = p_0(x - \xi; \epsilon, \tau)$ where

$$p_0(x; \epsilon, \tau) = (1 - \epsilon)\phi(x) + (\epsilon/\tau)\phi(x/\tau).$$

Moreover, all the distributions in the family $\mathcal{P}_0 = \{P_{0,\epsilon,\tau} : \epsilon \in [0, 1], \tau > 0\}$ are *symmetric about 0*. Here are plots of these densities for $\tau = 3$ and $\epsilon = 0, .2, .4, .6, .8, 1$; the second plot is of the densities p_0 for $\epsilon = .2$ and $\tau = 1.5, 4, 8, 16, 32, 64$.

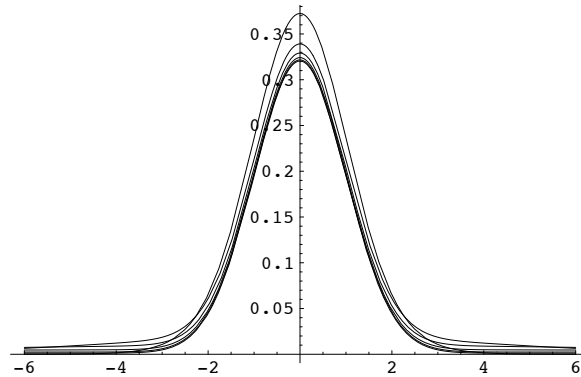


Figure 1: Densities $p(x)$ for $\xi = 0$, $\tau = 3$, $\epsilon = 0, .2, .4, .6, .8, 1.0$

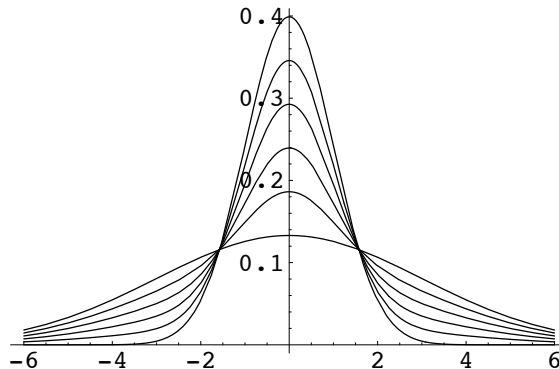


Figure 2: Densities $p(x)$ for $\xi = 0$, $\epsilon = .2$, $\tau = 1.5, 4, 8, 16, 32, 64$

Since $\phi'(x) = -x\phi(x)$, the score functions for ϵ , ξ , and τ are given by

$$\begin{aligned} \dot{l}_\xi(x) &= \frac{1}{p(x)} \left\{ (x - \xi)(1 - \epsilon)\phi(x - \xi) + \frac{x - \xi}{\tau^2} \frac{\epsilon}{\tau} \phi\left(\frac{x - \xi}{\tau}\right) \right\}, \\ \dot{l}_\epsilon(x) &= \frac{1}{p(x)} \left\{ \frac{1}{\tau} \phi\left(\frac{x - \xi}{\tau}\right) - \phi(x - \xi) \right\}, \\ \dot{l}_\tau(x) &= \frac{1}{p(x)} \frac{\epsilon}{\tau^2} \phi\left(\frac{x - \xi}{\tau}\right) \left\{ \left(\frac{x - \xi}{\tau}\right)^2 - 1 \right\}. \end{aligned}$$

Thus with $\theta \equiv (\xi, \epsilon, \tau)$ and \dot{l}_θ , the information matrix $I(\xi, \epsilon, \tau) = I(\theta)$ is given by

$$\begin{aligned} I(\theta) &= E_\theta \{ \dot{l}_\theta(X) \dot{l}_\theta(X)^T \} \\ &= \left(E_\theta \{ \dot{l}_i(X) \dot{l}_j(X) \} \right). \end{aligned}$$

Hence it becomes clear that all of the elements of $I(\epsilon, \xi, \tau)$ are constant functions of ξ ; hence it suffices to compute the information matrix for $\xi = 0$. Thus we take $\xi = 0$ in the rest of the argument. Now note that the scores for ϵ and τ are even functions of x : $\dot{l}_\epsilon(-x) = \dot{l}_\epsilon(x)$ and similarly for \dot{l}_τ . On the other hand, the score function for ξ is an odd function of x : $\dot{l}_\xi(-x) = -\dot{l}_\xi(x)$. It follows that

$$E_\theta \{ \dot{l}_\xi(X) \dot{l}_\epsilon(X) \} = 0, \quad \text{and} \quad E_\theta \{ \dot{l}_\xi(X) \dot{l}_\tau(X) \} = 0.$$

Thus the only non-zero entry off the diagonal is $I_{23}(\theta) = I_{\epsilon, \tau}(\theta)$. I do not know any ‘‘closed form’’ results for $I_{11}(\theta)$, $I_{22}(\theta)$, $I_{33}(\theta)$, or $I_{23}(\theta)$, but it is not hard to compute them as functions of $\theta = (\epsilon, \xi, \tau)$ especially since they depend only on (ϵ, τ) . See Figures 3-5 below.

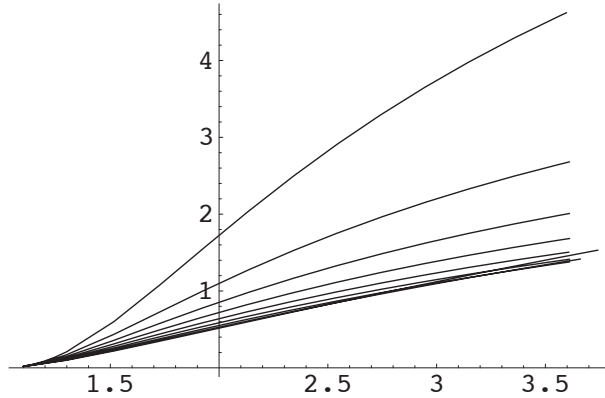


Figure 3: Information for ϵ as a function of τ for $\epsilon = .1, .2, .3, \dots, .9$

- Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \leq m < k$. Let $\dot{\mathbf{1}} = \dot{\mathbf{1}}_\theta = (\dot{\mathbf{1}}_1, \dot{\mathbf{1}}_2)$ be the corresponding partition of the (vector of) scores $\dot{\mathbf{1}}$, and, with $\tilde{\mathbf{1}} \equiv I^{-1}(\theta)\dot{\mathbf{1}}$, the *efficient influence function* for θ , let $\tilde{\mathbf{1}} = (\tilde{\mathbf{1}}_1, \tilde{\mathbf{1}}_2)$ be the corresponding partition of $\tilde{\mathbf{1}}$. In both

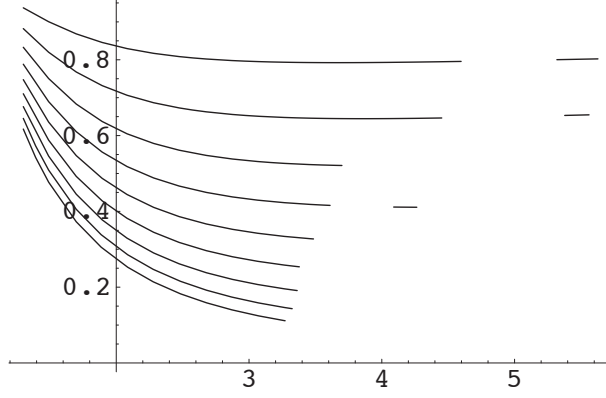


Figure 4: Information for ξ as a function of τ for $\epsilon = .1, .2, .3, \dots, .9$

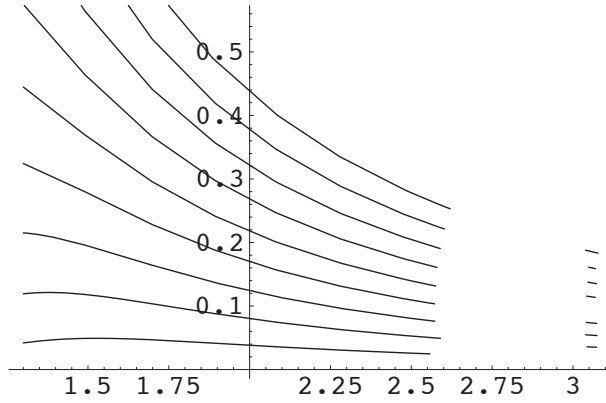


Figure 5: Information for τ as a function of τ for $\epsilon = .1, .2, .3, \dots, .9$

cases, $\dot{\mathbf{i}}_1, \tilde{\mathbf{i}}_1$ are m -vectors of functions, and $\dot{\mathbf{i}}_2, \tilde{\mathbf{i}}_2$ are $k - m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where I_{11} is $m \times m$, I_{12} is $m \times (k - m)$, I_{21} is $(k - m) \times m$, I_{22} is $(k - m) \times (k - m)$. Also write $I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}$.

(a) Verify that:

$$I^{11} = I_{11.2}^{-1} \text{ where } I_{11.2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21}, \quad I^{22} = I_{22.1}^{-1} \text{ where } I_{22.1} \equiv I_{22} - I_{21}I_{11}^{-1}I_{12},$$

$$I^{12} = -I_{11.2}^{-1}I_{12}I_{22}^{-1}, \quad \text{and } I^{21} = -I_{22.1}^{-1}I_{21}I_{11}^{-1}.$$

This amounts to formulas (4) and (5) of section 3.2, page 19.

(b) Verify that

$$\tilde{\mathbf{i}}_1 = I^{11}\dot{\mathbf{i}}_1 + I^{12}\dot{\mathbf{i}}_2 = I_{11.2}^{-1}(\dot{\mathbf{i}}_1 - I_{12}I_{22}^{-1}\dot{\mathbf{i}}_2), \quad \text{and}$$

$$\tilde{\mathbf{i}}_2 = I^{21}\dot{\mathbf{i}}_1 + I^{22}\dot{\mathbf{i}}_2 = I_{22.1}^{-1}(\dot{\mathbf{i}}_2 - I_{21}I_{11}^{-1}\dot{\mathbf{i}}_1).$$

(c) Verify that $\tilde{\mathbf{i}}_1 = I_{11}^{-1}\dot{\mathbf{i}}_1 - I_{11}^{-1}I_{12}I_{22}^{-1}\dot{\mathbf{i}}_2$ and hence that $I_{11.2}^{-1} = I_{11}^{-1} + I_{11}^{-1}I_{12}I_{22}^{-1}I_{21}I_{11}^{-1}$. This amounts to (15) and (16) of section 3.2, page 21.

Solution: (a) This is just block inversion/multiplication of matrices:

$$\begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} = \begin{pmatrix} I_{11.2}^{-1} & -I_{11.2}^{-1}I_{12}I_{22}^{-1} \\ -I_{22.1}^{-1}I_{21}I_{11}^{-1} & I_{22.1}^{-1} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} I_{11.2}^{-1}(I_{11} - I_{12}I_{22}^{-1}I_{21}) & I_{11.2}^{-1}(I_{12} - I_{12}) \\ I_{22.1}^{-1}(-I_{21} + I_{21}) & I_{22.1}^{-1}(-I_{21}I_{11}I_{12} + I_{22}) \end{pmatrix} \\
&= \begin{pmatrix} Ident & 0 \\ 0 & Ident \end{pmatrix} = Identity.
\end{aligned}$$

by using the definition of $I_{11.2}$ and $I_{22.1}$.

(b) This follows immediately from the formulas for I^{11} and I^{12} by just plugging into the formula $\tilde{\mathbf{I}}_1 = I^{11}\dot{\mathbf{I}}_1 + I^{12}\dot{\mathbf{I}}_2$ for $\tilde{\mathbf{I}}_1$:

$$\begin{aligned}
\tilde{\mathbf{I}}_1 &= I_{11.2}^{-1}\dot{\mathbf{I}}_1 - I_{11.2}^{-1}I_{12}I_{22}^{-1}\dot{\mathbf{I}}_2 \\
&= I_{11.2}^{-1}(\dot{\mathbf{I}}_1 - I_{12}I_{22}^{-1}\dot{\mathbf{I}}_2) = I_{11.2}^{-1}\mathbf{I}_1^*.
\end{aligned}$$

(c) As in the chapter 3 notes, page 21,

$$\begin{aligned}
\tilde{\mathbf{I}}_1 + I_{11}^{-1}\tilde{\mathbf{I}}_2 &= I^{11}\dot{\mathbf{I}}_1 + I^{12}\dot{\mathbf{I}}_2 + I_{11}^{-1}I_{12}(I^{21}\dot{\mathbf{I}}_1 + I^{22}\dot{\mathbf{I}}_2) \\
&= I_{11}^{-1} \left\{ (I_{11}I^{11} + I_{12}I^{21})\dot{\mathbf{I}}_1 + (I_{11}I^{21} + I_{12}I^{22})\dot{\mathbf{I}}_2 \right\} \\
&= I_{11}^{-1}\dot{\mathbf{I}}_1.
\end{aligned}$$

Rearranging yields the claimed identity:

$$\tilde{\mathbf{I}}_2 = I_{11}^{-1}\dot{\mathbf{I}}_1 - I_{11}^{-1}I_{12}\tilde{\mathbf{I}}_2.$$

Since $\tilde{\mathbf{I}}_2$ is orthogonal to $[\dot{\mathbf{I}}_1]$ (the linear span of the components of $\dot{\mathbf{I}}_1$), it follows that

$$\begin{aligned}
I_{11.2}^{-1} &= E(\tilde{\mathbf{I}}_1\tilde{\mathbf{I}}_1^T) = E(I_{11}^{-1}\dot{\mathbf{I}}_1 - I_{11}^{-1}I_{12}\tilde{\mathbf{I}}_2)^{\otimes 2} \\
&= I_{11}^{-1} + I_{11}^{-1}I_{12}I_{22.1}^{-1}I_{21}I_{11}^{-1}.
\end{aligned}$$

4. Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty) \equiv \Theta.$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0 ; here $F_\theta(x) \equiv P_\theta(X \leq x)$.

(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page 127, Table 6.1

(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.

(iii) Find the efficient influence functions for estimation of q_A and q_B .

(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators \bar{X}_n and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.

Solution: For the $\text{Gamma}(\alpha, \beta)$ parametrized *my way*:

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x).$$

Thus

$$\log p_\theta(x) = (\alpha - 1) \log x + \alpha \log \beta - \log \Gamma(\alpha) - \beta x,$$

and hence

$$\begin{aligned} \dot{l}_\alpha(x) &= \log x + \log \beta - \frac{\Gamma'}{\Gamma}(\alpha) = \log(\beta x) - \psi(\alpha), \\ \dot{l}_\beta(x) &= \frac{\alpha}{\beta} - x \end{aligned}$$

Furthermore,

$$\begin{aligned} \ddot{l}_{\alpha\alpha}(x) &= -\psi'(\alpha), \\ \ddot{l}_{\alpha\beta}(x) &= \frac{1}{\beta} = \ddot{l}_{\beta\alpha}(x), \\ \ddot{l}_{\beta\beta}(x) &= -\frac{\alpha}{\beta^2}. \end{aligned}$$

Hence

$$I(\theta) = \begin{pmatrix} \psi'(\alpha) & -1/\beta \\ -1/\beta & \alpha/\beta^2 \end{pmatrix}.$$

(ii). Now $q_A(\theta) = \alpha/\beta$, and

$$\begin{aligned} q_B(\theta) = P_\theta(X \leq x_0) &= \int_0^{x_0} \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} dx \\ &= \int_0^{\beta x_0} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t} dt \\ &\equiv \frac{\Gamma(\alpha, \beta x_0)}{\Gamma(\alpha)} \end{aligned}$$

where $\Gamma(\alpha, y)$ is the incomplete gamma function; note that $\Gamma(\alpha, \infty) = \Gamma(\alpha)$. Therefore

$$\begin{aligned} \dot{q}_A^T(\theta) &= \left(\frac{\partial}{\partial \alpha} q_A, \frac{\partial}{\partial \beta} q_B \right) = \left(\frac{1}{\beta}, -\frac{\alpha}{\beta^2} \right) = \frac{1}{\beta} \left(1, -\frac{\alpha}{\beta} \right) \\ &= \text{Cov}_\theta(X - E_\theta(X), \dot{l}_\theta^T(X)), \end{aligned}$$

while, with

$$\psi(\alpha, y) \equiv \frac{\partial}{\partial \alpha} \log \Gamma(\alpha, y) \equiv \Gamma'(\alpha, y)/\Gamma(\alpha, y),$$

$$\begin{aligned} \dot{q}_B^T(\theta) &= \left(\frac{\Gamma'(\alpha, \beta x_0)}{\Gamma(\alpha)} - \frac{\Gamma(\alpha, \beta x_0)\Gamma'(\alpha)}{\Gamma^2(\alpha)}, \frac{(\beta x_0)^{\alpha-1} e^{-\beta x_0} x_0}{\Gamma(\alpha) \beta} \right) \\ &= \left(\frac{\Gamma(\alpha, \beta x_0)}{\Gamma(\alpha)} \{ \psi(\alpha, \beta x_0) - \psi(\alpha) \}, \frac{x_0}{\beta} p_\theta(x_0) \right) \\ &= (q_B(\theta) \{ \psi(\alpha, \beta x_0) - \psi(\alpha) \}, \frac{x_0}{\beta} p_\theta(x_0)) \\ &= \text{Cov}_\theta[(1_{[0, x_0]}(X) - F_\theta(x_0)), \dot{l}_\theta^T]. \end{aligned}$$

(iii). The scores are given by

$$\dot{l}_\theta(x) = \begin{pmatrix} \dot{l}_\alpha(x) \\ \dot{l}_\beta(x) \end{pmatrix} = \begin{pmatrix} \log(\beta x) - \psi(\alpha) \\ \frac{\alpha}{\beta} - x \end{pmatrix}$$

and the information matrix is

$$I(\theta) = \begin{pmatrix} \psi'(\alpha) & -1/\beta \\ -1/\beta & \alpha/\beta^2 \end{pmatrix}.$$

Thus

$$I^{-1}(\theta) = \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1},$$

and the efficient influence function for estimation of q_A is

$$\begin{aligned} \tilde{l}_A &= \dot{q}_A(\theta)^T I^{-1}(\theta) \dot{l}_\theta \\ &= \frac{1}{\beta} \left(1, -\frac{\alpha}{\beta}\right) \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1} \begin{pmatrix} \log(\beta x) - \psi(\alpha) \\ \frac{\alpha}{\beta} - x \end{pmatrix} \\ &= \frac{\beta}{\alpha\psi'(\alpha) - 1} \left\{ 0 \cdot (\log(\beta x) - \psi(\alpha)) + \left(\frac{1}{\beta} - \frac{\alpha}{\beta}\psi'(\alpha)\right) \left(\frac{\alpha}{\beta} - x\right) \right\} \\ &= \left(x - \frac{\alpha}{\beta}\right). \end{aligned}$$

Note that $X - E_\theta(X) \in [\dot{l}_\theta] = \dot{\mathcal{P}}$; in fact, $X - E_\theta(X) = -\dot{l}_\beta(X)$.

Similarly, $\tilde{l}_B(x) = \dot{q}_B(\theta) I^{-1}(\theta) \dot{l}_\theta(x)$; unfortunately, this does not simplify much, largely due to the fact that $1_{[0, x_0]}(X) - F_\theta(x_0) \notin [\dot{l}_\theta] = \dot{\mathcal{P}}$.

(iii) The information bound for estimation of q_A is

$$\begin{aligned} I^{-1}(P|q_A, \mathcal{P}) &= \dot{q}_A^T I^{-1}(\theta) \dot{q}_A \\ &= \frac{1}{\beta} \left(1, -\frac{\alpha}{\beta}\right) \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1} \begin{pmatrix} 1 \\ -\alpha/\beta \end{pmatrix} \frac{1}{\beta} \\ &= \frac{\alpha}{\beta^2} = \text{Var}_\theta(X). \end{aligned}$$

Similarly,

$$I^{-1}(P|q_B, \mathcal{P}) = \dot{q}_B^T I^{-1}(\theta) \dot{q}_B,$$

which does not simplify appreciably because $1_{[0, x_0]}(X) - F_\theta(x_0) \notin [\dot{l}_\theta] = \dot{\mathcal{P}}$. However, since we know that $\tilde{l}_B = \Pi(1_{[0, x_0]}(x) - F_\theta(x_0) | \dot{\mathcal{P}})$, it follows easily that

$$I^{-1}(P|q_B, \mathcal{P}) < E_\theta(1_{[0, x_0]}(X) - F_\theta(x_0))^2 = F_\theta(x_0)(1 - F_\theta(x_0));$$

i.e. it is possible to improve on the natural nonparametric estimator $\mathbb{F}_n(x_0)$ of $q_B(\theta) = F_\theta(x_0)$ when the model holds.