Statistics 581, Problem Set 2 Solutions
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1. (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W, distribution function F_W, and inverse distribution function F_W^{-1} explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(-\pi/2, \pi/2)$. [Note that $g(\theta) = \tan(\theta)$ is a periodic function of θ with period π: $g(\theta) = g(\theta + \pi)$ for all θ. Thus the inverse function $g^{-1}(\theta) = \arctan(\theta)$ is usually taken to be the inverse of g restricted to $\theta \in (-\pi/2, \pi/2)$ where $\tan(\theta)$ is strictly increasing, negative for $-\pi/2 < \theta < 0$ and positive for $0 < \theta < \pi/2$.]
(c) Use the results of (a) and (b) to show (using Theorem 2.4.1, Chapter 2 notes, page 23) how to use two independent Uniform(0, 1) random variables U and V to generate two standard normal random variables.

Solution: (a) If $W \sim \chi_2^2 = \text{Gamma}(1, 1/2)$, the density function is given by $f_W(w) = (1/2)e^{-w/2}1_{[0,\infty)}(w)$; i.e. $W \sim \text{Exponential}(1/2)$. Hence the distribution function is $F_W(w) = 1 - \exp(-w/2)$ for $w \geq 0$, and the inverse distribution function is $F_W^{-1}(u) = -2\log(1-u)$.
(b) The joint density of (X, Y) is given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi}\exp(-(x^2 + y^2)/2) \qquad \text{for}(x, y) \in R^2\,.$$

Moreover, $x = r\cos(\theta)$ and $y = r\sin(\theta)$ uniquely for $r \in (0, \infty)$ and $\theta \in (-\pi, \pi]$. However, the inverse function $(x, y) \mapsto (\sqrt{x^2 + y^2}, \arctan(y/x)) \equiv (r, \theta)$ maps both the half planes $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $\{(x, y) \in \mathbb{R}^2 : x < 0\}$ into $\{(r, \theta) : r > 0, -\pi/2 < \theta < \pi/2)\}$. Hence both of these half planes contribute separately to the joint density of (r, θ). Since the Jacobian of the transformation $(r, \theta) \mapsto (x(r, \theta), y(r, \theta))$ is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \left| \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \right| = r\cos^2(\theta) + r\sin^2(\theta) = r\,,$$

we find that the joint density of (R, Θ) is given by

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= f_{X,Y}(r\cos(\theta), r\sin(\theta))1_{(0,\infty)}(r\cos(\theta)) \\ &\quad + f_{X,Y}(r\cos(\theta + \pi), r\sin(\theta + \pi))1_{(-\infty,0)}(r\cos(\theta + \pi)) \\ &= \frac{1}{2\pi}\exp(-r^2/2)r + \frac{1}{2\pi}\exp(-r^2/2)r \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\exp(-r^2/2)r \qquad \text{on} \ \ (0,\infty)\times(-\pi/2,\pi/2] \\
&= r\exp(-r^2/2)\cdot\frac{1}{\pi} = f_R(r)f_\Theta(\theta)\,.
\end{aligned}
$$

Thus R and Θ are independent with densities $f_R(r) = r\exp(-r^2/2)1_{(0,\infty)}(r)$ and $f_\Theta(\theta) = \pi^{-1}1_{(-\pi/2,\pi/2]}(\theta)$. Note that the distribution function of R is given by

$$F_R(r) = \int_0^r f_R(y)dy = \int_0^r y\exp(-y^2/2)dy = 1-\exp(-r^2/2)\,.$$

It follows easily from this that

$$F_{R^2}(x) = P(R^2\le x) = P(R\le\sqrt{x}) = 1-\exp(-x/2)$$

for $x\in[0,\infty)$; i.e. $R^2\sim$ Exponential$(1/2) =$ Gamma$(1,1/2) = \chi_2^2$.
(c) If U and V are independent Uniform$(0,1)$ random variables, we can use the inverse transformation to first obtain

$$R^2 \equiv F_{\chi_2^2}^{-1}(U) = -2\log(1-U)\sim\chi_2^2 \qquad \text{and} \qquad \Theta\equiv 2\pi V\sim \text{ Uniform}(0,2\pi)$$

note that R^2 and Θ are independent by independence of U and V. Then in view of (b)

$$(X,Y)\equiv(R\cos(\Theta),R\sin(\Theta))\sim N_2(0,I)\,.$$

2. (See Ferguson, ACILST, p. 94-100.) Suppose that $X_1,X_2,\ldots$ are iid Exponential(λ). Let $M_n\equiv\min_{1\le i\le n}X_i$ and $T_n\equiv\max_{1\le i\le n}X_i$.
(a) Show that $nM_n \overset{d}{=}$ exponential(λ).
(b) Show that $T_n-(1/\lambda)\log n\to_d (1/\lambda)T$ where T has the double exponential extreme value distribution function given by $P(T\le x) = \exp(-\exp(-x))$.
(c) Now suppose that $X_1,\ldots,X_n$ are i.i.d. on $\mathbb{R}^+$ with distribution function F satisfying $F(0)=0$ and $0<F'(0)<\infty$; here $F'(0)$ is the right-derivative of F at 0:

$$\lim_{x\searrow 0}\frac{F(x)-F(0)}{x} = F'(0).$$

Show that $nM_n\to_d$ exponential $(F'(0))$.

Solution: (a) Now

$$\begin{aligned} P(nM_n > x) &= P\left(\min_{1\le k\le n} X_k > \frac{x}{n}\right) \\ &= P\left(X_1 > \frac{x}{n}, \ldots, X_n > \frac{x}{n}\right) \\ &= P\left(X_1 > \frac{x}{n}\right) \cdots P\left(X_n > \frac{x}{n}\right) \\ &= P(X_1 > x/n)^n = \{\exp(-\lambda x/n)\}^n \\ &= \exp(-\lambda x). \end{aligned}$$

Hence $nM_n \stackrel{d}{=} \text{exponential}(\lambda)$.
(b) For the maximum T_n,

$$\begin{aligned} P(T_n - (1/\lambda)\log n \le x) &= P(\max_{1\le i\le n} X_i \le x + (1/\lambda)\log n) \\ &= P(X_1 \le x + (1/\lambda)\log n)^n \\ &= (1 - \exp(-\lambda x - \log n))^n \\ &= \left(1 - \frac{\exp(-\lambda x)}{n}\right)^n \to \exp(-\exp(-\lambda x)) \\ &= P(T \le \lambda x) \end{aligned}$$

where T has the extreme value distribution function $P(T \le x) = \exp(-\exp(-x)$ for $x \in \mathbb{R}$.
(c) In this case, the reasoning is much the same in part (a) at the beginning:

$$\begin{aligned} P(nM_n > x) &= P(X_1 > x/n)^n \\ &= \left(1 - \frac{nF(x/n)}{n}\right)^n \end{aligned} \tag{1}$$

Since F has a derivative F' (from the right) at 0 and $F(0) = 0$ we have

$$nF(x/n) = \frac{F(x/n) - F(0)}{x/n} \cdot x \to F'(0)x \qquad \text{as} \;\; n \to \infty.$$

Thus the expression on the right side of (1) converges to $\exp(-F'(0)x)$ as $n \to \infty$; i.e. $nM_n \to$ exponential $(F'(0))$.

3. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{E[(Y - E(Y|X))^2|X]\} + E\{[E(Y|X) - E(Y)]^2\}.$$

(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).
(d) Show that

$$\frac{\chi_n^2(\delta) - (n + \delta)}{\sqrt{2n + 4\delta}} \to_d N(0, 1)$$

as either $n \to \infty$ or $\delta \to \infty$.

Solution: (a) We compute directly:

$$\begin{aligned} Var(Y) &= E[Y - E(Y)]^2 = E[Y - E(Y|X) + E(Y|X) - E(Y)]^2 \\ &= E[Y - E(Y|X)]^2 + 2E[(Y - E(Y|X)][E(Y|X) - E(Y)] \\ &\qquad + E[E(Y|X) - E(Y)]^2 \\ &= E\{E\{[Y - E(Y|X)]^2|X\}\} + 0 + Var[E(Y|X)] \\ &= E\{Var[Y|X]\} + Var[E(Y|X)] \end{aligned}$$

since, by computing conditionally,

$$\begin{aligned} &E[(Y - E(Y|X)][E(Y|X) - E(Y)] \\ &\quad = E\{E\{[(Y - E(Y|X)][E(Y|X) - E(Y)]|X\}\} \\ &\quad = E\{[E(Y|X) - E(Y)]E\{[Y - E(Y|X)]|X\}\} \\ &\quad = E\{[E(Y|X) - E(Y)]\{E(Y|X) - E(Y|X)\}\} \\ &\quad = E\{[E(Y|X) - E(Y)] \cdot 0\} \\ &\quad = 0. \end{aligned}$$

(b) A geometric interpretation of (a) is that $Y - E(Y|X)$ is orthogonal to $E(Y|X) - E(Y)$ in $L_2(\Omega, \mathcal{A}, P) = L_2(P)$, thus the identity in (a) can be intepreted as a "pythagorean theorem". Also note that $Y - E(Y|X)$ is orthogonal to any function $g(X)$: much as in the last part of (a)

$$\begin{aligned}
&E[(Y - E(Y|X)]g(X) \\
&\quad = E\{E\{[(Y - E(Y|X)]g(X)|X\}\} \\
&\quad = E\{g(X)E\{[Y - E(Y|X)]|X\}\} \\
&\quad = E\{g(X)\{E(Y|X) - E(Y|X)\}\} \\
&\quad = E\{g(X) \cdot 0\} \\
&\quad = 0\,.
\end{aligned}$$

(c) Now $(Y|K) \sim \chi^2_{2K+n}$ where $K \sim \text{Poisson}(\delta/2)$, so

$$E(Y) = E\{E(Y|K)\} = E\{2K + n\} = n + 2(\delta/2) = n + \delta\,.$$

Furthermore, using part (a) we get

$$\begin{aligned}
Var(Y) &= E\{Var(Y|K)\} + Var\{E(Y|K)\} \\
&= E\{2(2K + n)\} + Var\{2K + n\} \\
&= 4(\delta/2) + 2n + 4(\delta/2) \\
&= 2n + 4\delta\,.
\end{aligned}$$

(d) First suppose that $n \to \infty$. (d) First note that if $Y \sim \chi^2_n(\delta)$, then $Y =_d (Z_1 + \sqrt{\delta})^2 + Z_2^2 + \cdots + Z_n^2$ where $Z_1, Z_2, \ldots, Z_n$ are independent, $Z_1, \ldots, Z_n \sim N(0,1)$. We can write, with $T_i \equiv Z_i^2 - 1$ for $i = 2, \ldots, n$ (having mean 0 and variance 2),

$$\begin{aligned}
\frac{\chi^2_n(\delta) - (n + \delta)}{\sqrt{2n + 4\delta}} &=_d \frac{(Z_1 + \sqrt{\delta})^2 - (1 + \delta)) + (Z_2^2 - 1) + \cdots + (Z_n^2 - 1) - \delta}{\sqrt{2n + 4\delta}} \\
&= \frac{(Z_1^2 - 1) + \cdots + (Z_n^2 - 1)}{\sqrt{2n + 4\delta}} + \frac{2\sqrt{\delta}Z_1}{\sqrt{2n + 4\delta}} \\
&= \frac{T_1 + \cdots + T_n}{\sqrt{2n}} \frac{\sqrt{2n}}{\sqrt{2n + 4\delta}} + \frac{2\sqrt{\delta}Z_1}{\sqrt{2n + 4\delta}}
\end{aligned}$$

where, as $n \to \infty$,

$$\frac{2\sqrt{\delta}Z_1}{\sqrt{2n + 4\delta}} \to_p 0; \qquad \text{and} \qquad \frac{T_1 + \cdots + T_n}{\sqrt{2n}} \to_d N(0,1)$$

by the CLT. Hence by Slutsky's theorem

$$\frac{\chi_n^2(\delta) - (n+\delta)}{\sqrt{2n+4\delta}} \to_d N(0,1) \cdot 1 + 0 = N(0,1) \qquad \text{as} \;\; n \to \infty\,.$$

If n is fixed and $\delta \to \infty$,

$$\frac{T_1 + \cdots + T_n}{\sqrt{2n+4\delta}} \to_p 0$$

while

$$\frac{2\sqrt{\delta} Z_1}{\sqrt{2n+4\delta}} \to_d Z_1 \sim N(0,1)\,.$$

Hence the desired conclusion follows from Slutsky's theorem (Proposition 2.2.9).

4. Suppose that:
(i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) $\Sigma^2 = \Sigma$ (so Σ is a projection matrix);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

Solution: See Ferguson, ACILST, page 63 (and page 57). Find Γ orthogonal so that $\Gamma'\Sigma\Gamma = D$ where D is diagonal. Now $\Gamma\Gamma' = I$, so if $\Sigma^2 = \Sigma$ we have $D^2 = \Gamma'\Sigma\Gamma\Gamma'\Sigma\Gamma = \Gamma'\Sigma^2\Gamma = \Gamma'\Sigma\Gamma = D$ and conversely. Moreover, since Σ is of rank k, D is of rank k, and this together with $D^2 = D$ implies that D has k 1's on the diagonal and $n-k$ 0's. Without loss, assume that Γ has been chosen so that the k ones occur in the first r positions of the diagonal matrix D; thus

$$D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is $k \times k$. Moreover, note that

$$D\Gamma'\mu = \Gamma'\Sigma\Gamma\Gamma'\mu = \Gamma'\Sigma\mu = \Gamma'\mu,$$

and this implies that the last $n-k$ components of $\Gamma'\mu$ are all zero. Now let $Y = \Gamma'X$ (much as in the proof of theorem 1.3.2 of the notes). Then $Y \sim N_n(\Gamma'\mu, D)$, $Y'Y = X'\Gamma\Gamma'X = X'X$, and by (d) of page 16, section 1.3, $X'X = Y'Y \sim \chi_k^2(\delta)$ where $\delta = (\Gamma'\mu)'(\Gamma'\mu) = \mu'\Gamma\Gamma'\mu = \mu'\mu$.

5. Ferguson, ACILST, #4, page 6:
(a) Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.
(b) Give an example of a sequence of random variables X_n such that $X_n \to_p 0$ but $X_n \to_{a.s.} 0$ fails.

Solution: (a) If $X_n = a_n$ with probability p_n and $X_n = 0$ with probability $1 - p_n$, then $E(X_n) = a_n p_n$ and $E(X_n^2) = a_n^2 p_n = 1$ if $p_n = 1/a_n^2$. Then $E(X_n) = a_n/a_n^2 = 1/a_n \to 0$ if $a_n \to \infty$. Ferguson's solution on page 173 takes $a_n = n$; the same holds for any sequence $a_n \to \infty$.
(b) Let $U \sim$ Uniform$(0, 1)$, and set $X_n = n^{\alpha} 1_{[0,1/n]}(U)$. Then $EX_n = n^{\alpha} n^{-1} \to 0$ if $\alpha < 1$, while $EX_n^2 = n^{2\alpha} n^{-1} \to \infty$ if $\alpha > 1/2$. Thus the required convergences hold for all $1/2 < \alpha < 1$.
(c) Let $U \sim$ Uniform$(0, 1)$. The "dancing functions" are defined by $X_{n,k} = 1_{[(k-1)/2^n, k/2^n)}(U)$, $k = 1, \ldots, 2^n$, $n = 1, 2, \ldots$. Let $\{Y_m\}_{m \geq 1}$ be defined by $Y_m = X_{n,k}$ if $m = (\sum_{j=1}^{n} 2^j) + k = 2^{n+1} - 2 + k$ with $1 \leq k \leq 2^n$. Then for $\epsilon \in (0, 1)$,

$$P(|Y_m| > \epsilon) = P(|X_{n,k}| > \epsilon) = 2^{-n} \to 0$$

so $Y_m \to_p 0$, but for every $U(\omega) \in (0, 1)$ we have $Y_m(\omega) = 1$ for infinitely many m's and also $Y_m(\omega) = 0$ for infinitely many m's. Hence

$$0 = \liminf Y_m < \limsup Y_m = 1 \qquad a.s.$$

and it follows that Y_m does not converge to 0 almost surely.