Statistics 581, Problem Set 9

Wellner; 11/22/2017

Reading: Chapter 4, Sections 1-2;
Ferguson, ACLST, Chapters 18- 20, pages 119-125, 133-139; Chapter 22, pages 144-150;
vdV, Asymp. Statist., pages 41 - 75; Sections 5.1 - 5.7.
Due: Wednesday, November 29, 2017.

1. (a) Ferguson, ACILST, problem 17.2, page 117: I would suggest modifying Ferguson's definition of the density to:

$$p_\theta(x) \equiv f(x|\theta) = 2\left\{ \frac{x}{\theta} 1_{[0,\theta]}(x) + \frac{1-x}{1-\theta} 1_{(\theta,1]}(x) \right\}.$$

   (a-i) Let $0 = X_{(0)}, X_{(1)}, \ldots, X_{(n)}, X_{(n+1)} = 1$ denote the order statistics. Show that for $X_{(k)} \le \theta \le X_{(k+1)}$ the likelihood function is decreasing if $\theta < k/n$ and increasing if $\theta > k/n$.
   (a-ii) Conclude that the maximum likelihood estimate is equal to one of the $X_{(k)}$ for which $(k-1)/n \le X_{(k)} \le k/n$. In fact, the likelihood function has a local maximum at each such $X_{(k)}$.
   (b) Do our hypotheses A0-A2 of Chapter 4, Section 1, hold in this example?
   (c) Compute $K(P_{\theta_0}, P_\theta)$ where P_θ has density as given in this problem.
   (d) Do our hypotheses A3 and A4 hold in this example? Why or why not?
   (e) Does there exist an estimator $\overline{\theta}_n$ of θ which is $n^{1/2}-$consistent?
   In connection with this problem note that the likelihood function $L(\theta)$ given on Ferguson's page 215 is *not correct*: the formula there should be replaced by

$$L_n(\theta|\underline{X}) \;=\; \left(\frac{2}{\theta}\right)^k \prod_{j=1}^{k} X_{(j)} \cdot \left(\frac{2}{1-\theta}\right)^{n-k} \prod_{j=k+1}^{n} (1 - X_{(j)}) \;\text{ if }\; X_{(k)} \le \theta < X_{(k+1)}.$$

2. (a) Lehmann and Casella, problem 6.3.1, page 501: Let X have the binomial distribution $Binomial(n,p)$, $0 \le p \le 1$. Determine the MLE of p by:
   (i) The usual calculus method of determining the maximum of a function.
   (ii) Showing that $p^x(1-p)^{n-x} \le (x/n)^x((n-x)/n)^{n-x}$.
   (b) Lehmann and Casella, problem 6.3.2, page 501: In the preceding problem, show that the MLE does not exist when p is restricted to $0 < p < 1$ and when $X = 0$ or $X = n$.
   (c) Lehmann and Casella, problem 6.3.4, page 501: Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, 1)$ with $\theta > 0$. Show that the MLE is $\overline{X}_n$ when $\overline{X}_n > 0$ and does not exist when $\overline{X}_n \le 0$.

3. (a) vdV, Chapter 7, problem 1, page 106: Show that the family of Poisson distributions

$$\mathcal{P} = \{P_\theta : \; p_\theta(k) = \exp(-\theta)\theta^k/k!, \; \theta > 0\}$$

satisfies the conditions of Lemma 7.6 (vdV, page 95).
(b) If $X_1, \ldots, X_n$ are i.i.d. $P_{\theta_0} \in \mathcal{P}$, what can you conclude about

$$\log \prod_{i=1}^{n} \frac{p_{\theta_0 + hn^{-1/2}}}{p_{\theta_0}}(X_i)?$$

(c) Can you prove the same results as in (b) by a direct argument?

4. (a) Show that the Laplace location family is differentiable in quadratic mean. What is the consequence of this for the behavior of the local log-likelihood ratios? What is the resulting information for the location parameter θ?
(b) Apply the methods of section 3.5 to show that with $\theta_0 \in \mathbb{R}$ is fixed and $\theta_n = \theta_0 + n^{-1/2}h$, then for any estimator T_n of θ we have

$$\liminf_{n\to\infty} \inf_{T_n} \max\{E_{\theta_n} n|T_n - \theta_n|^2,\ E_{\theta_0} n|T_n - \theta_0|^2\} \geq cI(\theta_0)^{-1}$$

for some choice of h and an absolute constant c.

5. Suppose that $X_1, \ldots, X_n$ are i.i.d. log-normal(μ, σ^2) with density

$$p_\theta(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right) 1_{(0,\infty)}(x).$$

Here $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^+ \equiv (-\infty, \infty) \times (0, \infty)$.
(a) Find the MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ of $\theta = (\mu, \sigma^2)$.
(b) Show that $\log X \overset{d}{=} \mu + \sigma Z \sim N(\mu, \sigma^2)$ where $Z \sim N(0, 1)$.
(c) Suppose that $\nu(P_\theta) = q(\theta) = E_\theta(X)$. Express $q(\theta)$ explicitly as a function of θ.
(d) Suggest a natural nonparametric estimator $\overline{\nu}_n$ of $E_\theta(X)$.
(e) Find the asymptotic variance of $\sqrt{n}(\overline{\nu}_n - \nu(P_\theta))$ for the estimator $\overline{\nu}_n$ you proposed in (d).
(f) What is the MLE $\hat{\nu}_n$ of $\nu = \nu(P_\theta)$ assuming that the log-normal model is true? What do our results in chapter 3 say about the asymptotic distribution of $\sqrt{n}(\hat{\nu}_n - \nu(P_\theta))$ (assuming that the model holds)?
(g) Compare the variances you found in (e) and (f). Which estimator do you prefer if the log-normal model holds?

6. **Optional bonus problem 1:** Lehmann and Casella, TPE, problem 6.3.22, page 503, reworded as follows. (In other words, prove (vi) of theorem 1.2, pages 5-6, chapter 4 notes). Suppose that $X_1, \ldots, X_n$ are i.i.d. with density p_θ, $\theta \in \Theta \subset R^k$, satisfying the hypotheses of theorem 4.1, page 463 (the Cramér conditions given in (A) - (D) on pages 462-463). Show that the following Local Asymptotic Normality (LAN) result holds for the (local) log- likelihood ratios: with

$$L_n(\theta) \equiv \log(\prod_{i=1}^{n} p_\theta(X_i)) = \sum_{i=1}^{n} \log p_\theta(X_i),$$

for a fixed $\theta_0 \in \Theta$,

$$L_n(\theta_0 + n^{-1/2}\underline{t}) - L_n(\theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underline{t}^T \dot{\underline{l}}_\theta(X_i) - \frac{1}{2}\underline{t}^T I(\theta_0)\underline{t} + o_p(1)$$

$$
\begin{aligned}
&\to_d \quad N(0, t^T I(\theta_0) t) - \frac{1}{2} t^T I(\theta_0) t \\
&\overset{d}{=} \quad N(-\sigma^2/2, \sigma^2)
\end{aligned}
$$

under P_{θ_0} where $\sigma^2 \equiv \underline{t}^T I(\theta_0) \underline{t}$. (The convergence in the last display actually holds under the considerably weaker hypothesis of Hellinger differentiability of p_θ at θ_0, as stated in Corollary 3 of section 3.3, page 28, of the Chapter 3 notes.)