Statistics 581, Problem Set 8

Wellner; 11/15/2017

Reading: Chapter 3, Sections 3-5; start reading Chapter 4 (to be handed out on Monday, November 20);
Ferguson, ACILST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;
vdV, Asymp. Statist., pages 85 - 97; Sections 6.1 - 6.2; 7.1 - 7.2.

Due: Wednesday, November 22, 2017.

1. (a) Show that if $\theta_n = cn^{-1/2}$ and T_n is the Hodges super-efficient estimator discussed in class, then the sequence $\{\sqrt{n}(T_n - \theta_n)\}$ is uniformly square-integrable.
(b) Let $R_n(\theta) \equiv nE_\theta(T_n - \theta)^2$ where T_n is the Hodges superefficient estimator as in Example 3.3.1 (so $T_n = \delta_n$ of Example 2.5, Lehmann and Casella pages 440 - 443). Show that $R_n(n^{-1/4}) \to \infty$ as $n \to \infty$.

2. Lehmann and Casella, Problem 2.13, page 501.
Let $b_n(\theta) = E_\theta(T_n) - \theta$ be the bias of Hodges estimator T_n.
(a) Show that
$$b_n(\theta) = \frac{-(1-a)}{\sqrt{n}} \int_{-n^{1/4}}^{n^{1/4}} x\phi(x - \sqrt{n}\theta)dx.$$
(b) Show that $b_n'(\theta) \to 0$ for any $\theta \neq 0$ and $b_n'(0) \to 1 - \alpha$.
(c) Use (b) to explain how the Hodges estimator T_n can violate $V^2(\theta)$ without violating (Cramér-Rao) information inequality.

3. Suppose that $Z \sim N(0,1)$ and, for $\mu \in R$ and $\sigma > 0$, that $X = \mu + \sigma Z \sim P_{\mu,\sigma} = N(\mu, \sigma^2)$.
(a) Compute the likelihood ratio
$$\frac{dP_{\mu,\sigma}}{dP_{0,\sigma}}(x) = \frac{\sigma^{-1}\phi((x-\mu)/\sigma)}{\sigma^{-1}\phi(x/\sigma)} \qquad \text{and} \qquad Y \equiv \log \frac{dP_{\mu,\sigma}}{dP_{0,\sigma}}(X).$$
What is the distribution of Y under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$?
(b) Plot the function $l(\mu; X) \equiv \log(dP_{\mu,\sigma}/dP_{0,\sigma})(X)$ as a function of μ.
(c) Find the maximum value of the function $l(\mu; X)$ in (b) (as a function of μ) and the value of $\mu \equiv \hat{\mu}$ which achieves the maximum.
(d) What is the distribution of $\hat{\mu}$ under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$? What is the distribution of $l(\hat{\mu}; X)$ under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$?

4. Suppose that $(T|Z) \sim \text{Weibull}(\lambda^{-1}e^{-\gamma Z}, \beta)$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ. Thus the conditional cumulative hazard function $\Lambda(t|z)$ is given by
$$\Lambda_{\gamma,\lambda,\beta}(t|z) = (\lambda e^{\gamma Z} t)^\beta = \lambda^\beta e^{\beta\gamma Z} t^\beta$$
and hence
$$\lambda_{\gamma,\lambda,\beta}(t|z) = \lambda^\beta e^{\beta\gamma Z} \beta t^{\beta-1}.$$

(Recall that $\lambda(t) = f(t)/(1 - F(t))$ and

$$\Lambda(t) \equiv \int_0^t \lambda(s)ds = \int_0^t (1 - F(s))^{-1} dF(s) = -\log(1 - F(t))$$

if F is continuous.) Thus it makes sense to re-parametrize by defining $\theta_1 \equiv \beta\gamma$ (this is the parameter of interest since it reflects the effect of the covariate Z), $\theta_2 \equiv \lambda^\beta$, and $\theta_3 \equiv \beta$. This yields

$$\lambda_\theta(t|z) = \theta_3 \theta_2 \exp(\theta_1 z) t^{\theta_3 - 1}$$

You may assume that

$$a(z) \equiv (\partial/\partial\eta) \log g_\eta(z)$$

exists and $E\{a^2(Z)\} < \infty$. Thus Z is a "covariate" or "predictor variable", θ_1 is a "regression parameter" which affects the intensity of the (conditionally) Weibull variable T, and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ where $\theta_4 \equiv \eta$.

(a) Derive the joint density $p_\theta(t, z)$ of (T, Z) for the re-parametrized model.

(b) Find the information matrix for θ. What does the structure of this matrix say about the effect of $\eta = \theta_4$ being known or unknown about the estimation of $\theta_1, \theta_2, \theta_3$?

(c) Find the information and information bound for θ_1 if the parameters θ_2 and θ_3 are known.

(d) What is the information bound for θ_1 if just θ_3 is known to be equal to 1?

(e) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_3 is known.

(f) Find the information $I_{11\cdot(2,3)}$ and information bound for θ_1 if the parameters θ_2 and θ_3 are unknown. (Here both θ_2 and θ_3 are in "the second block".)

(g) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_2 and θ_3 are unknown.

(h) Specialize the calculations in (d) - (g) to the case when $Z \sim$ Bernoulli(θ_4) and compare the information bounds.

5. **Optional bonus problem 1:** Suppose that $X \sim F_\theta =$ exponential(θ) with density $f_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$ and $Y \sim G_\eta$ independent of X with densities $\{g_\eta : \eta \in R^+\}$, a regular parametric model on $(0, \infty)$. Consider the following three scenarios for observation of X or functions of X:
(a) Uncensored: we observe X and Y.
(b) Right-censored: we observe $T(X, Y) = (X \wedge Y, 1\{X \le Y\} \equiv (\min\{X, Y\}, 1\{X \le Y\}) \equiv (Z, \Delta)$.
(c) Interval-censored (case 1): we observe $S(X, Y) = (Y, 1\{X \le Y\}) \equiv (Y, \Delta)$.
In each of the three scenarios (a), (b), (c):
(i) Find the joint density of (X, Y) and joint distributions of $T(X, Y)$ and $S(X, Y)$.
(ii) Find the scores for θ and η. (Let $(\partial/\partial\eta) \log g_\eta(y) \equiv a(y)$ with $a \in L_2^0(G_\eta)$.)
(iii) Compute and compare $I_{X,Y}(\theta)$, $I_{T(X,Y)}(\theta)$, and $I_{S(X,Y)}(\theta)$. Make the comparisons in general and then explicitly by making one or more choices of the family $\{g_\eta\}$.

6. **Optional bonus problem 2:** Suppose that $X_1, \ldots, X_n$ are i.i.d. F on $\mathbb{R}$, and let $\mathbb{F}_n$ denote the empirical d.f. of the X_i's. Let Φ denote the standard normal distribution function, $\Phi(x) = \int_{-\infty}^{x} \phi(y)dy$ where $\phi(y) = (2\pi)^{-1/2}\exp(-y^2/2)$ is the standard normal density. Let $0 < a < 1$ and define a new estimator $\widetilde{F}_n$ of F by

$$\widetilde{F}_n(x) = \begin{cases} (1-a)\Phi(x) + a\mathbb{F}_n(x), & \text{if } \|\mathbb{F}_n - \Phi\|_\infty \le n^{-1/4}, \\ \mathbb{F}_n(x), & \text{if } \|\mathbb{F}_n - \Phi\|_\infty > n^{-1/4}. \end{cases}$$

(a) Find the limiting distribution of the process $\{\sqrt{n}(\widetilde{F}_n(x) - F(x)) : x \in \mathbb{R}\}$ when $F = \Phi$.
(b) Find the limiting distribution of the process $\{\sqrt{n}(\widetilde{F}_n(x) - F(x)) : x \in \mathbb{R}\}$ when $F \ne \Phi$.
(c) Show that $\widetilde{F}_n$ is not a regular estimator of F at $F = \Phi$ (in an appropriate sense to be defined), but that F is a regular estimator of F at any $F \ne \Phi$.