

Statistics 581, Problem Set 7

Wellner; 11/8/2017

Reading: Chapter 3, Sections 2-4;
 Ferguson, ACILST, Chapters 19-20, pages 126-139;
 vdV, Asymp. Statist., pages 108 - 119; Sections 8.1 - 8.7.

Due: Wednesday, November 15, 2017.

1. Consider the two parameter location-scale model

$$\mathcal{P} = \left\{ P_\theta : \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta \right\}$$

where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right),$$

and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ .

- (a) Calculate the information matrix $I(\theta)$ for θ .
- (b) For which of the densities in (a)-(e) of problem set #6, problem 3, is $I_{12}(\theta)$ not zero?

2. Lehmann and Casella, TPE, Problem 6.6, page 142. If P_θ is the probability measure with density p_θ with respect to Lebesgue measure on \mathbb{R} given by

$$p_\theta(x) = (1 - \epsilon)\phi(x - \xi) + \frac{\epsilon}{\tau}\phi\left(\frac{x - \xi}{\tau}\right)$$

where ϕ is the standard normal density and $\theta = (\xi, \tau, \epsilon) \in \mathbb{R} \times \mathbb{R}^+ \times [0, 1]$, find the information matrix $I(\theta) = I(\xi, \tau, \epsilon)$.

3. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \leq m < k$. Let $\dot{\mathbf{i}} = \dot{\mathbf{i}} = (\dot{\mathbf{i}}_1, \dot{\mathbf{i}}_2)$ be the corresponding partition of the (vector of) scores $\dot{\mathbf{i}}$, and, with $\tilde{\mathbf{I}} \equiv I^{-1}(\theta)\dot{\mathbf{i}}$, the *efficient influence function* for θ , let $\tilde{\mathbf{I}} = (\tilde{\mathbf{I}}_1, \tilde{\mathbf{I}}_2)$ be the corresponding partition of $\tilde{\mathbf{I}}$. In both cases, $\dot{\mathbf{i}}_1, \tilde{\mathbf{I}}_1$ are m -vectors of functions, and $\dot{\mathbf{i}}_2, \tilde{\mathbf{I}}_2$ are $k - m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where I_{11} is $m \times m$, I_{12} is $m \times (k - m)$, I_{21} is $(k - m) \times m$, I_{22} is $(k - m) \times (k - m)$. Also write $I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}$.

- (a) Verify that:

$$I^{11} = I_{11.2}^{-1} \text{ where } I_{11.2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21}, \quad I^{22} = I_{22.1}^{-1} \text{ where } I_{22.1} \equiv I_{22} - I_{21}I_{11}^{-1}I_{12}, \\ I^{12} = -I_{11.2}^{-1}I_{12}I_{22}^{-1}, \text{ and } I^{21} = -I_{22.1}^{-1}I_{21}I_{11}^{-1}.$$

This amounts to formulas (4) and (5) of section 3.2, page 19.

(b) Verify that

$$\tilde{\mathbf{I}}_1 = I^{11}\mathbf{i}_1 + I^{12}\mathbf{i}_2 = I_{11.2}^{-1}(\mathbf{i}_1 - I_{12}I_{22}^{-1}\mathbf{i}_2), \text{ and}$$

$$\tilde{\mathbf{I}}_2 = I^{21}\mathbf{i}_1 + I^{22}\mathbf{i}_2 = I_{22.1}^{-1}(\mathbf{i}_2 - I_{21}I_{11}^{-1}\mathbf{i}_1).$$

(c) Verify that $\tilde{\mathbf{I}}_1 = I_{11}^{-1}\mathbf{i}_1 - I_{11}^{-1}I_{12}\tilde{\mathbf{I}}_2$ and hence that $I_{11.2}^{-1} = I_{11}^{-1} + I_{11}^{-1}I_{12}I_{22.1}^{-1}I_{21}I_{11}^{-1}$. This amounts to (15) and (16) of section 3.2, page 21.

4. Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) \mathbf{1}_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty) \equiv \Theta.$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0 ; here $F_\theta(x) \equiv P_\theta(X \leq x)$.

(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page 127, Table 6.1

(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.

(iii) Find the efficient influence functions for estimation of q_A and q_B .

(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators \bar{X}_n and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.

5. **Optional bonus problem:** [This is example 7.2.5 and 7.2.7 in Lehmann and Casella, TPE, section 6.2; also see problems 6.2.12 - 6.2.14, Lehmann and Casella, TPE, page 501.] Suppose that X_1, \dots, X_n are i.i.d. $N(\theta, 1)$ so $I(\theta) = 1$. Let $0 < a < 1$ and define $T_n \equiv \bar{X}_n \mathbf{1}_{\{|\bar{X}_n| \geq n^{-1/4}\}} + a\bar{X}_n \mathbf{1}_{\{|\bar{X}_n| < n^{-1/4}\}}$. This is Hodges superefficient estimator of θ .

(a) Show that $\sqrt{n}(T_n - \theta) \rightarrow_d N(0, V(\theta))$ where

$$V(\theta) = \begin{cases} 1, & \theta \neq 0 \\ a^2, & \theta = 0 \end{cases}$$

(b) Show that T_n is *not* a regular estimator of θ at $\theta = 0$, but that it is regular at every $\theta \neq 0$. If $\theta_n = t/\sqrt{n}$, find the limiting distribution of $\sqrt{n}(T_n - \theta_n)$ under P_{θ_n} .

C. For $\theta_n = t/\sqrt{n}$ show that

$$R_n(\theta_n) = nE_{\theta_n}(T_n - \theta_n)^2 \rightarrow E(aZ + t(a-1))^2 = a^2 + t^2(1-a)^2$$

where $Z \sim N(0, 1)$. This is *larger* than 1 if $t^2 > (1+a)/(1-a)$, and hence superefficiency also entails *worse* risks in a local neighborhood of the point(s) where the asymptotic variance is smaller.