Statistics 581, Problem Set 2
Wellner; 10/4/2017

Reading:
Course Notes: Chapter 1, especially pages 12 - 19; start reading Chapter 2.
Ferguson, pages 8-35.
van der Vaart, Sections 2.2 - 2.9 (pages 12 - 24).
Due: Wednesday, October 11, 2017.

1. (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W, distribution function F_W, and inverse distribution function F_W^{-1} explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(-\pi/2, \pi/2)$. [Note that $g(\theta) = \tan(\theta)$ is a periodic function of θ with period π: $g(\theta) = g(\theta + \pi)$ for all θ. Thus the inverse function $g^{-1}(\theta) = \arctan(\theta)$ is usually taken to be the inverse of g restricted to $\theta \in (-\pi/2, \pi/2)$ where $\tan(\theta)$ is strictly increasing, negative for $-\pi/2 < \theta < 0$ and positive for $0 < \theta < \pi/2$.]
(c) Use the results of (a) and (b) to show (using Theorem 2.4.1, Chapter 2 notes, page 23) how to use two independent $\text{Uniform}(0, 1)$ random variables U and V to generate two standard normal random variables.

2. (See Ferguson, ACILST, p. 94-100.) Suppose that $X_1, X_2, \ldots$ are iid Exponential(λ). Let $M_n \equiv \min_{1 \le i \le n} X_i$ and $T_n \equiv \max_{1 \le i \le n} X_i$.
(a) Show that $nM_n \stackrel{d}{=} \text{exponential}(\lambda)$.
(b) Show that $T_n - (1/\lambda)\log n \to_d (1/\lambda)T$ where T has the double exponential extreme value distribution function given by $P(T \le x) = \exp(-\exp(-x))$.
(c) Now suppose that $X_1, \ldots, X_n$ are iid with distribution function F satisfying $0 < F'(0) < \infty$; here $F'(0)$ is the right-derivative of F at 0:
$$\lim_{x \searrow 0} \frac{F(x) - F(0)}{x} = F'(0).$$
Show that $nM_n \to_d$ exponential $(F'(0))$.

3. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that
$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}\,;$$
i.e.
$$E(Y - EY)^2 = E\{E[(Y - E(Y|X))^2|X]\} + E\{[E(Y|X) - E(Y)]^2\}\,.$$

(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).
(d) Show that

$$\frac{\chi_n^2(\delta) - (n+\delta)}{\sqrt{2n+4\delta}} \to_d N(0,1)$$

as either $n \to \infty$ or $\delta \to \infty$.

4. Suppose that:
(i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) $\Sigma^2 = \Sigma$ (so Σ is a projection matrix);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

5. Ferguson, ACILST, #4, page 6:
(a) Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.
(b) Give an example of a sequence of random variables X_n such that $X_n \to_p 0$ but $X_n \to_{a.s.} 0$ fails.

6. **Optional Bonus Problem 1:**
For $\theta > 0$, $\theta \neq 1$, let

$$C_\theta(u,v) \equiv \log\{1 + (\theta^u - 1)(\theta^v - 1)/(\theta - 1)\}/\log\theta.$$

(i) Find the density $c_\theta(u,v)$ corresponding to the distribution function $C_\theta(u,v)$.
(ii) Show that C_θ is a distribution function on $[0,1]^2$ with uniform marginal distributions.
(iii) Show that $C_\theta(u,v) \to u \cdot v$ for $0 < u, v < 1$ as $\theta \to 1$.
(iv) Show that if $(U,V) \sim C_\theta$ then $(1-U, 1-V) \stackrel{d}{=} (U,V)$.

7. **Optional Bonus Problem 2:**
Wellner 581 Course Notes, Chapter 1, Exercise 4.1, page 19.