Statistics 581, Problem Set 1

Wellner; 9/27/2017

Reading: Ferguson, ACILST, Chapter 1, pages 3-7;
Van der Vaart, Chapter 2, pages 2 - 12; skim Chapter 0; read Chapter 1.
Due: Wednesday, October 4, 2017.

1. Ferguson, ACILST, #2, page 6: Suppose that X_n is uniformly distributed on the set of points $\{1/n, 2/n, \ldots, 1\}$. Show that $X_n \rightarrow_d X$ where the distribution of X is Uniform$(0,1)$. Does $X_n \rightarrow_p X$?

2. (Continuation of the previous problem). Now suppose that $U \sim$ Uniform$(0,1)$ and for each $n \geq 1$ define $V_n \equiv \sum_{j=1}^n (j/n) 1_{((j-1)/n, j/n]}(U)$.
   (a) Show that $V_n \stackrel{d}{=} X_n$ where X_n is as in problem 2.
   (b) Show that $V_n \rightarrow_p U$.

3. Suppose that X has an inverse power distribution with distribution function $F_\alpha(x) = 1 - x^{-\alpha}$ for $x \in [1, \infty)$ where $\alpha > 0$.
   (a) Show that $E(X) = \alpha/(\alpha - 1)$ for $\alpha > 1$ and
   $Var(X) = \alpha/((\alpha-2)(\alpha-1)^2)$ for $\alpha > 2$.
   (b) Let $Y_\alpha = (\alpha - 1)X - \alpha$ (so that Y_α has mean 0 and variance converging to 1 as $\alpha \rightarrow \infty$). Show that $Y_\alpha \rightarrow_d Y_\infty$ for some random variable Y_∞ and find the distribution function of Y_∞.

4. (Lehmann and Casella, TPE, problems 1.2 and 1.3, page 62.)
   (a) Let $X_1, \ldots, X_n$ be uncorrelated random variables with common expectation θ and variance σ^2. Show that, among all linear estimators $\sum \alpha_i X_i$ of θ satisfying $\sum \alpha_i = 1$, the mean $\overline{X}_n$ has the smallest variance.
   (b) In the preceding problem, minimize the variance of
   $\sum \alpha_i X_i$ ($\sum \alpha_i = 1$)
   (i) When the variance of X_i is σ^2/c_i (c_i known).
   (ii) When the X_i have common variance σ^2 but are correlated with common correlation coefficient ρ.

5. (a) First suppose that X has distribution function F on $\mathbb{R}^+ = [0, \infty)$ and Y with distribution function G on $\mathbb{R}^+$ are independent random variables. Find the joint distribution of $Z = \min\{X, Y\}$ and $W = 1\{X \leq Y\}$. That is, find $P(Z \leq z, W = 1)$ and $P(Z \leq z, W = 0)$ in terms of F and G.

(b) Show that if $F = \text{exponential}(\lambda)$ and $G = \text{exponential}(\mu)$, then Z and W are independent.

6. **Bonus problem 1:** (van der Vaart (1998), page 24, problem 1.) Suppose that X_n has a $t-$distribution with n degrees of freedom. Show that $X_n \rightarrow_d X \sim N(0,1)$ as $n \rightarrow \infty$.

7. **Bonus problem 2:** (van der Vaart (1998), page 24, problem 2.) Does it follow immediately from the result of the previous problem that $E(X_n^p) \rightarrow E(X^p)$ for every non-negative integer p? Is this true?

8. **Bonus problem 3:** Let $\mathcal{X} = (0,1)$, $\mathcal{Y} = (0,1)$, both equipped with the Borel sets and Lebesgue measure. Let

$$g(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad \text{for} \quad (x,y) \in (0,1) \times (0,1)\,.$$

Show that:
(a) $\int_0^1 (\int_0^1 g(x,y)dy)dx = \pi/4$.
(b) $\int_0^1 (\int_0^1 g(x,y)dx)dy = -\pi/4$.
(c) Why does Fubini's theorem fail here?