Statistics 581, Problem Set 7 Solutions

Wellner; 11/12/2014

1. Consider the two parameter location-scale model

$$\mathcal{P} = \left\{ P_\theta : \ \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta \right\}$$

where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right),$$

and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ.

(a) Calculate the information matrix $I(\theta)$ for θ.

(b) For which of the densities in (a)-(e) of problem set #6, problem 3, is $I_{12}(\theta)$ not zero?

Solution: (a) This follows Lehmann and Casella, Example 6.5, pages 126-127. The score for location is

$$\begin{aligned} \dot{\mathbf{l}}_1(x) &= \frac{\partial}{\partial\theta_1} \log\left\{\frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right)\right\} \\ &= -\frac{f'}{f}\left(\frac{x-\theta_1}{\theta_2}\right)\frac{1}{\theta_2} \end{aligned}$$

and the score for scale is

$$\begin{aligned} \dot{\mathbf{l}}_2(x) &= \frac{\partial}{\partial\theta_2} \log\left\{\frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right)\right\} \\ &= -\frac{1}{\theta_2} - \frac{f'}{f}\left(\frac{x-\theta_1}{\theta_2}\right)\frac{(x-\theta_1)}{\theta_2^2} \end{aligned}$$

Thus we compute

$$\begin{aligned} I_{11}(\theta) &= E\dot{\mathbf{l}}_1^2(X) = \frac{1}{\theta_2^2}\int \left(\frac{f'}{f}\left(\frac{x-\theta_1}{\theta_2}\right)\right)^2 \frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right) dx \\ &= \frac{1}{\theta_2^2}\int \left(\frac{f'}{f}(y)\right)^2 f(y)dy \equiv \frac{1}{\theta_2^2} I_{f,loc}, \\ I_{22}(\theta) &= E\dot{\mathbf{l}}_2^2(X) = \frac{1}{\theta_2^2}\int \left(-1 - \frac{(x-\theta_1)}{\theta_2}\frac{f'}{f}\left(\frac{x-\theta_1}{\theta_2}\right)\right)^2 \frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right) dx \\ &= \frac{1}{\theta_2^2}\int \left(-1 - y\frac{f'}{f}(y)\right)^2 f(y)dy \equiv \frac{1}{\theta_2^2} I_{f,scale}, \\ I_{12}(\theta) &= E\dot{\mathbf{l}}_1(X)\dot{\mathbf{l}}_2(X) = I_{21}(\theta) \\ &= \frac{1}{\theta_2^2}\int \left(-\frac{f'}{f}\left(\frac{x-\theta_1}{\theta_2}\right)\right)\left(-1 - \frac{(x-\theta_1)}{\theta_2}\frac{f'}{f}\left(\frac{x-\theta_1}{\theta_2}\right)\right)\frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right) dx \\ &= \frac{1}{\theta_2^2}\int y\left(\frac{f'}{f}(y)\right)^2 f(y)dy \equiv \frac{1}{\theta_2^2} I_{12,f}. \end{aligned}$$

(b) Note that $I_{12,f} = 0$ for all symmetric densities f since the score for location, $\dot{\mathbf{l}}_1(y) = -f'(y)/f(y)$ is odd $\dot{\mathbf{l}}_1(-y) = -\dot{\mathbf{l}}_1(y)$, while the score for scale, $\dot{\mathbf{l}}_2(y) = -1 - yf'(y)/f(y)$, is even ($\dot{\mathbf{l}}_2(-y) = \dot{\mathbf{l}}_2(y)$), and f is even. Thus $I_{12}(\theta) = 0$ for cases (a)-(d) in problem 3, problem set #6, while $I_{12}(\theta) = \theta_2^{-2} I_{12,f} \neq 0$ in case (e). I calculate

$$\begin{aligned} I_{12,f} &= \int y \left(\frac{f'}{f}(y) \right)^2 f(y) dy \\ &= \int_{-\infty}^{\infty} y(-1 + e^{-y})^2 e^{-y} \exp(-e^{-y}) dy \\ &= -\int_0^{\infty} (\log v)(v-1)^2 e^{-v} dv = -(1-\gamma) \end{aligned}$$

where γ is Euler's constant. In fact this is related to I_{12} that we calculated already for the Weibull family in Example 3.2.5 in the notes and in the first display at the top of page 2 of the Handout on Gamma, Digama, and Polygamma. Note that from problem 4, problem set #6, we also have

$I_{f,loc} = 1$	for $f = \phi$, the N(0,1) density,
$I_{f,loc} = 1/3$	for $f =$ the logistic density,
$I_{f,loc} = 1$	for $f =$ the Laplace density,
$I_{f,loc} = (k+1)/(k+2)$	for $f =$ the t_k density,
$I_{f,loc} = 1$	for $f =$ the Gumbel or extreme value density.

On the other hand, we find that $I_{f,scale}$ is given the the various cases as follows: (a) For the normal density $f = \phi$, $I_{f,scale} = \int (x^2 - 1)\phi(x) dx = Var(Z^2) = 2$ where $Z \sim N(0,1)$.
(b) For the logistic density f, the information for scale is

$$\begin{aligned} I_{f,scale} &= \int_{-\infty}^{\infty} \left\{ x \left(\frac{1 - e^{-x}}{1 + e^{-x}} \right) - 1 \right\}^2 dF(x) \\ &= \int_{-\infty}^{\infty} \{ x(2F(x) - 1) - 1 \}^2 dF(x) = (3 + \pi^2)/9 \end{aligned}$$

as is given in detail by Johnson and Kotz (1970) and by deCani and Stine (1986).
(c) For the double exponential (or Laplace) density $-(f'/f)(x) = \text{sign}(x)$, so $I_{f,scale} = 1$.
(d) For the t_k density, by using a change of variables much as in the location case,

$$I_{f,scale} = \frac{2k}{k+3};$$

note that $I_{t_k,scale} = (2k)/(k+3) \to 2 = I_{N(0,1),scale}$ as $k \to \infty$.
(e) For the extreme value distribution, $F(x) = \exp(-\exp(-x))$, and therefore if $X \sim F$ the random variable $Y = \exp(-X) \sim$ exponential(1):

$$P(Y \geq y) = P(\exp(-X) \geq y) = P(X \leq -\log(y)) = \exp(-\exp(\log y)) = \exp(-y).$$

Since $-(f'/f)(x) = -1 + e^{-x}$, it is easily seen that

$$\begin{aligned} I_f &= E\{-1 - X(f'/f)(X)\}^2 = E\{(X(1 - e^{-X}) - 1\}^2 \\ &= E\{(\log Y)(Y-1) - 1\}^2 = \frac{\pi^2}{6} + (1-\gamma)^2 \end{aligned}$$

as in our information calculations for the Weibull family.

2. Lehmann and Casella, TPE, Problem 6.6, page 142. if $p(x) = (1-\epsilon)\phi(x-\xi) + (\epsilon/\tau)\phi((x-\xi)/\tau)$ where ϕ is the standard normal density, find $I(\epsilon, \xi, \tau)$.

   Solution: I would prefer to write $\theta = (\xi, \epsilon, \tau)$. The first thing to note about the model
$$\mathcal{P} = \{P_\theta : \ (dP_\theta/d\lambda)(x) = p(x;\theta) = p(x)\}$$
is that it is a location family: $p(x;\xi,\epsilon,\tau) = p_0(x-\xi;\epsilon,\tau)$ where
$$p_0(x;\epsilon,\tau) = (1-\epsilon)\phi(x) + (\epsilon/\tau)\phi(x/\tau).$$
Moreover, all the distributions in the family $\mathcal{P}_0 = \{P_{0,\epsilon,\tau} : \epsilon \in [0,1], \tau > 0\}$ are *symmetric about* 0. Here are plots of these densities for $\tau = 3$ and $\epsilon = 0, .2, .4, .6, .8, 1$; the second plot is of the densities p_0 for $\epsilon = .2$ and $\tau = 1.5, 4, 8, 16, 32, 64$.

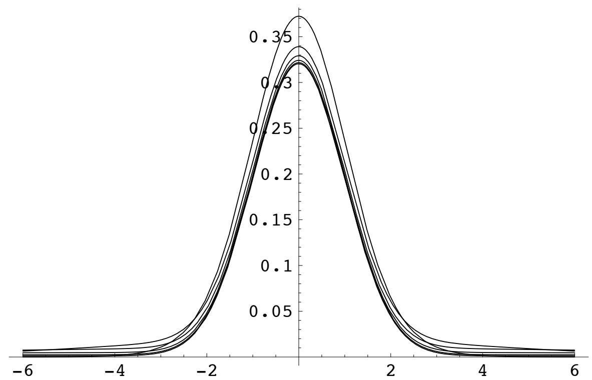

Figure 1: Densities $p(x)$ for $\xi = 0$, $\tau = 3$, $\epsilon = 0, .2, .4, .6, .8, 1.0$

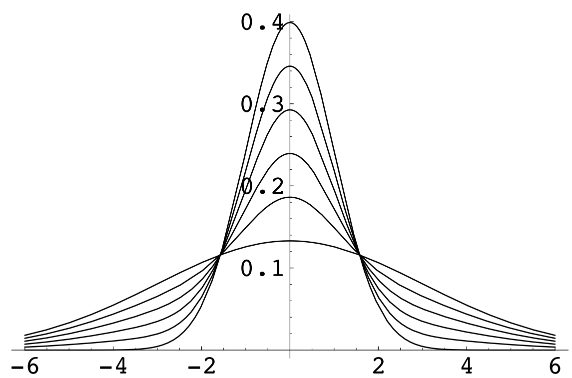

Figure 2: Densities $p(x)$ for $\xi = 0$, $\epsilon = .2$, $\tau = 1.5, 4, 8, 16, 32, 64$

Since $\phi'(x) = -x\phi(x)$, the score functions for ϵ, ξ, and τ are given by
$$\dot{l}_\xi(x) \quad = \quad \frac{1}{p(x)}\left\{(x-\xi)(1-\epsilon)\phi(x-\xi) + \frac{x-\xi}{\tau^2}\frac{\epsilon}{\tau}\phi\left(\frac{x-\xi}{\tau}\right)\right\},$$

$$\dot{l}_\epsilon(x) = \frac{1}{p(x)}\left\{\frac{1}{\tau}\phi\left(\frac{x-\xi}{\tau}\right) - \phi(x-\xi)\right\},$$
$$\dot{l}_\tau(x) = \frac{1}{p(x)}\frac{\epsilon}{\tau^2}\phi\left(\frac{x-\xi}{\tau}\right)\left\{\left(\frac{x-\xi}{\tau}\right)^2 - 1\right\}.$$

Thus with $\theta \equiv (\xi, \epsilon, \tau)$ and $\dot{\underline{l}}_\theta$, the information matrix $I(\xi, \epsilon, \tau) = I(\theta)$ is given by

$$\begin{aligned} I(\theta) &= E_\theta\{\dot{\underline{l}}_\theta(X)\dot{\underline{l}}_\theta(X)^T\} \\ &= \left(E_\theta(\dot{l}_i(X)\dot{l}_j(X))\right). \end{aligned}$$

Hence it becomes clear that all of the elements of $I(\epsilon, \xi, \tau)$ are constant functions of ξ; hence it suffices to compute the information matrix for $\xi = 0$. Thus we take $\xi = 0$ in the rest of the argument. Now note that the scores for ϵ and τ are even functions of x: $\dot{l}_\epsilon(-x) = \dot{l}_\epsilon(x)$ and similarly for $\dot{l}_\tau$. On the other hand, the score function for ξ is an odd function of x: $\dot{l}_\xi(-x) = -\dot{l}_\xi(x)$. It follows that

$$E_\theta\{\dot{l}_\xi(X)\dot{l}_\epsilon(X)\} = 0, \qquad \text{and} \qquad E_\theta\{\dot{l}_\xi(X)\dot{l}_\tau(X)\} = 0.$$

Thus the only non-zero entry off the diagonal is $I_{23}(\theta) = I_{\epsilon,\tau}(\theta)$. I do not know any "closed form" results for $I_{11}(\theta)$, $I_{22}(\theta)$, $I_{33}(\theta)$, or $I_{23}(\theta)$, but it is not hard to compute them as functions of $\theta = (\epsilon, \xi, \tau)$ especially since they depend only on (ϵ, τ). See Figures 3-5 below.

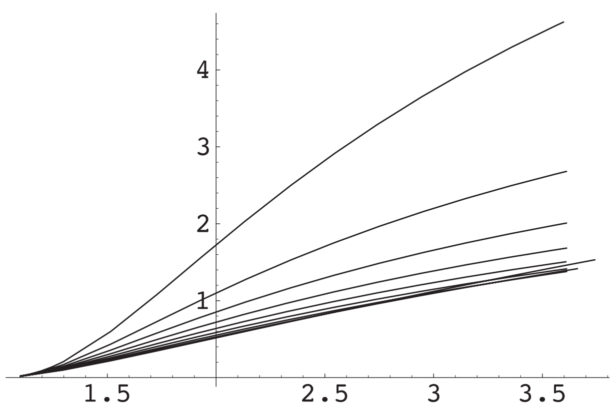

Figure 3: Information for ϵ as a function of τ for $\epsilon = .1, .2, .3, \ldots, .9$

3. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \le m < k$. Let $\dot{\mathbf{l}} = \dot{\mathbf{l}}_\theta = (\dot{\mathbf{l}}_1, \dot{\mathbf{l}}_2)$ be the corresponding partition of the (vector of) scores $\dot{\mathbf{l}}$, and, with $\widetilde{\mathbf{l}} \equiv I^{-1}(\theta)\dot{\mathbf{l}}$, the *efficient influence function* for θ, let $\widetilde{\mathbf{l}} = (\widetilde{\mathbf{l}}_1, \widetilde{\mathbf{l}}_2)$ be the corresponding partition of $\widetilde{\mathbf{l}}$. In both cases, $\dot{\mathbf{l}}_1$, $\widetilde{\mathbf{l}}_1$ are $m-$vectors of functions, and $\dot{\mathbf{l}}_2$, $\widetilde{\mathbf{l}}_2$ are $k-m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

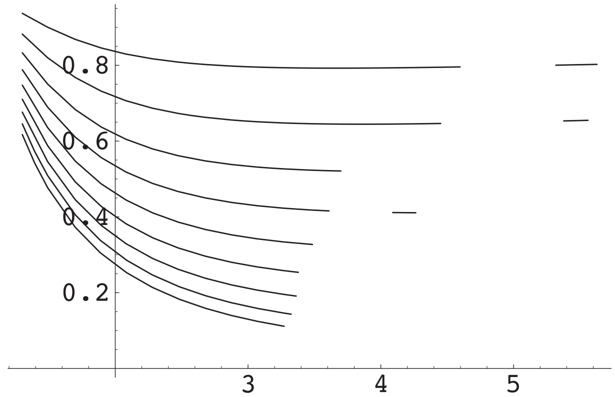

Figure 4: Information for ξ as a function of τ for $\epsilon = .1, .2, .3, \ldots, .9$

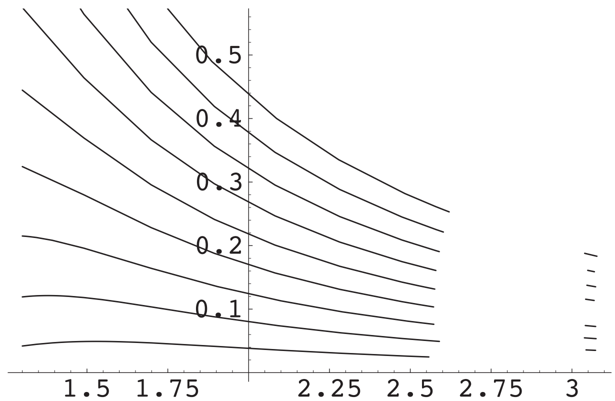

Figure 5: Information for τ as a function of τ for $\epsilon = .1, .2, .3, \ldots, .9$

where I_{11} is $m \times m$, I_{12} is $m \times (k-m)$, I_{21} is $(k-m) \times m$, I_{22} is $(k-m) \times (k-m)$. Also write $I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}$.

(a) Verify that:
$I^{11} = I_{11\cdot 2}^{-1}$ where $I_{11\cdot 2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21}$, $I^{22} = I_{22\cdot 1}^{-1}$ where $I_{22\cdot 1} \equiv I_{22} - I_{21}I_{11}^{-1}I_{12}$, $I^{12} = -I_{11\cdot 2}^{-1}I_{12}I_{22}^{-1}$, and $I^{21} = -I_{22\cdot 1}^{-1}I_{21}I_{11}^{-1}$.
This amounts to formulas (4) and (5) of section 3.2, page 19.

(b) Verify that
$\widetilde{\mathbf{l}}_1 = I^{11}\dot{\mathbf{l}}_1 + I^{12}\dot{\mathbf{l}}_2 = I_{11\cdot 2}^{-1}(\dot{\mathbf{l}}_1 - I_{12}I_{22}^{-1}\dot{\mathbf{l}}_2)$, and
$\widetilde{\mathbf{l}}_2 = I^{21}\dot{l}_1 + I^{22}\dot{\mathbf{l}}_2 = I_{22\cdot 1}^{-1}(\dot{\mathbf{l}}_2 - I_{21}I_{11}^{-1}\dot{\mathbf{l}}_1)$.

(c) Verify that $\widetilde{\mathbf{l}}_1 = I_{11}^{-1}\dot{\mathbf{l}}_1 - I_{11}^{-1}I_{12}\widetilde{\mathbf{l}}_2$ and hence that $I_{11\cdot 2}^{-1} = I_{11}^{-1} + I_{11}^{-1}I_{12}I_{22\cdot 1}^{-1}I_{21}I_{11}^{-1}$.
This amounts to (15) and (16) of section 3.2, page 21.

Solution: (a) This is just block inversion/multiplication of matrices:

$$\begin{aligned}
\begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix}\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} &= \begin{pmatrix} I_{11\cdot}^{-1} & -I_{11\cdot 2}^{-1}I_{21}I_{22}^{-1} \\ -I_{22\cdot 1}^{-1}I_{21}I_{11}^{-1} & I_{22\cdot 1} \end{pmatrix}\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \\
&= \begin{pmatrix} I_{11\cdot 2}^{-1}(I_{11} - I_{12}I_{22}^{-1}I_{21}) & I_{11\cdot 2}^{-1}(I_{12} - I_{12}) \\ I_{22\cdot 1}^{-1}(-I_{21} + I_{21}) & I_{22\cdot 1}^{-1}(-I_{21}I_{11}I_{12} + I_{22}) \end{pmatrix} \\
&= \begin{pmatrix} Ident & 0 \\ 0 & Ident \end{pmatrix} = Identity.
\end{aligned}$$

by using the definition of $I_{11\cdot 2}$ and $I_{22\cdot 1}$.

(b) This follows immediately from the formulas for I^{11} and I^{12} by just plugging into the formula $\tilde{\mathbf{l}}_1 = I^{11}\dot{\mathbf{l}}_1 + I^{12}\dot{\mathbf{l}}_2$ for $\tilde{\mathbf{l}}_1$:

$$
\begin{aligned}
\tilde{\mathbf{l}}_1 &= I_{11\cdot 2}^{-1}\dot{\mathbf{l}}_1 - I_{11\cdot 2}^{-1} I_{12} I_{22}^{-1}\dot{\mathbf{l}}_2 \\
&= I_{11\cdot 2}^{-1}(\dot{\mathbf{l}}_1 - I_{12} I_{22}^{-1}\dot{\mathbf{l}}_2) = I_{11\cdot 2}^{-1}\mathbf{l}_1^* .
\end{aligned}
$$

(c) As in the chapter 3 notes, page 21,

$$
\begin{aligned}
\tilde{\mathbf{l}}_1 + I_{11}^{-1}\tilde{\mathbf{l}}_2 &= I^{11}\dot{\mathbf{l}}_1 + I^{12}\dot{\mathbf{l}}_2 + I_{11}^{-1} I_{12}(I^{21}\dot{\mathbf{l}}_1 + I^{22}\dot{\mathbf{l}}_2) \\
&= I_{11}^{-1}\left\{(I_{11}I^{11} + I_{12}I^{21})\dot{\mathbf{l}}_1 + (I_{11}I^{21} + I_{12}I^{22})\dot{\mathbf{l}}_2\right\} \\
&= I_{11}^{-1}\dot{\mathbf{l}}_1.
\end{aligned}
$$

Rearranging yields the claimed identity:

$$
\tilde{\mathbf{l}}_2 = I_{11}^{-1}\dot{\mathbf{l}}_1 - I_{11}^{-1} I_{12}\tilde{\mathbf{l}}_2.
$$

Since $\tilde{\mathbf{l}}_2$ is orthogonal to $[\dot{\mathbf{l}}_1]$ (the linear span of the components of $\dot{\mathbf{l}}_1$), it follows that

$$
\begin{aligned}
I_{11\cdot 2}^{-1} &= E(\tilde{\mathbf{l}}_1\tilde{\mathbf{l}}_1^T) = E(I_{11}^{-1}\dot{\mathbf{l}}_1 - I_{11}^{-1} I_{12}\tilde{l}_2)^{\otimes 2} \\
&= I_{11}^{-1} + I_{11}^{-1} I_{12} I_{22\cdot 1}^{-1} I_{21} I_{11}^{-1}.
\end{aligned}
$$

4. **Optional bonus problem:** Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$
p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0,\infty)\times(0,\infty) \equiv \Theta .
$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0; here $F_\theta(x) \equiv P_\theta(X \le x)$.
(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page 127, Table 6.1
(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
(iii) Find the efficient influence functions for estimation of q_A and q_B.
(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators $\overline{X}_n$ and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.

Solution: For the Gamma(α, β) parametrized *my* way:

$$
p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-\beta x) 1_{(0,\infty)}(x).
$$

Thus

$$
\log p_\theta(x) = (\alpha - 1)\log x + \alpha\log\beta - \log\Gamma(\alpha) - \beta x,
$$

and hence

$$
\begin{aligned}
\dot{l}_\alpha(x) &= \log x + \log\beta - \frac{\Gamma'}{\Gamma}(\alpha) = \log(\beta x) - \psi(\alpha), \\
\dot{l}_\beta(x) &= \frac{\alpha}{\beta} - x
\end{aligned}
$$

Furthermore,

$$\begin{aligned}
\ddot{l}_{\alpha\alpha}(x) &= -\psi'(\alpha), \\
\ddot{l}_{\alpha\beta}(x) &= \frac{1}{\beta} = \ddot{l}_{\beta\alpha}(x), \\
\ddot{l}_{\beta\beta}(x) &= -\frac{\alpha}{\beta^2}.
\end{aligned}$$

Hence

$$I(\theta) = \begin{pmatrix} \psi'(\alpha) & -1/\beta \\ -1/\beta & \alpha/\beta^2 \end{pmatrix}.$$

(ii). Now $q_A(\theta) = \alpha/\beta$, and

$$\begin{aligned}
q_B(\theta) = P_\theta(X \le x_0) &= \int_0^{x_0} \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x}\, dx \\
&= \int_0^{\beta x_0} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t}\, dt \\
&\equiv \frac{\Gamma(\alpha, \beta x_0)}{\Gamma(\alpha)}
\end{aligned}$$

where $\Gamma(\alpha, y)$ is the incomplete gamma function; note that $\Gamma(\alpha, \infty) = \Gamma(\alpha)$. Therefore

$$\begin{aligned}
\dot{q}_A^T(\theta) = \left(\frac{\partial}{\partial\alpha} q_A, \frac{\partial}{\partial\beta} q_B \right) &= (\frac{1}{\beta}, -\frac{\alpha}{\beta^2}) = \frac{1}{\beta}(1, -\frac{\alpha}{\beta}) \\
&= \mathrm{Cov}_\theta(X - E_\theta(X), \dot{l}_\theta^T(X)),
\end{aligned}$$

while, with

$$\psi(\alpha, y) \equiv \frac{\partial}{\partial\alpha} \log \Gamma(\alpha, y) \equiv \Gamma'(\alpha, y)/\Gamma(\alpha, y)\,,$$

$$\begin{aligned}
\dot{q}_B^T(\theta) &= \left(\frac{\Gamma'(\alpha, \beta x_0)}{\Gamma(\alpha)} - \frac{\Gamma(\alpha, \beta x_0)\Gamma'(\alpha)}{\Gamma^2(\alpha)}, \frac{(\beta x_0)^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta x_0} \frac{x_0}{\beta} \beta \right) \\
&= \left(\frac{\Gamma(\alpha, \beta x_0)}{\Gamma(\alpha)} \{\psi(\alpha, \beta x_0) - \psi(\alpha)\}, \frac{x_0}{\beta} p_\theta(x_0) \right) \\
&= (q_B(\theta)\{\psi(\alpha, \beta x_0) - \psi(\alpha)\}, \frac{x_0}{\beta} p_\theta(x_0)) \\
&= \mathrm{Cov}_\theta[(1_{[0,x_0]}(X) - F_\theta(x_0)), \dot{l}_\theta^T]\,.
\end{aligned}$$

(iii). The scores are given by

$$\dot{l}_\theta(x) = \begin{pmatrix} \dot{l}_\alpha(x) \\ \dot{l}_\beta(x) \end{pmatrix} = \begin{pmatrix} \log(\beta x) - \psi(\alpha) \\ \frac{\alpha}{\beta} - x \end{pmatrix}$$

and the information matrix is

$$I(\theta) = \begin{pmatrix} \psi'(\alpha) & -1/\beta \\ -1/\beta & \alpha/\beta^2 \end{pmatrix}.$$

Thus

$$I^{-1}(\theta) = \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1},$$

and the efficient influence function for estimation of q_A is

$$\begin{aligned}
\tilde{l}_A &= \dot{q}_A(\theta)^T I^{-1}(\theta)\dot{l}_\theta \\
&= \frac{1}{\beta}(1, -\frac{\alpha}{\beta}) \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1} \begin{pmatrix} \log(\beta x) - \psi(\alpha) \\ \frac{\alpha}{\beta} - x \end{pmatrix} \\
&= \frac{\beta}{\alpha\psi'(\alpha) - 1}\{0 \cdot (\log(\beta x) - \psi(\alpha)) + (\frac{1}{\beta} - \frac{\alpha}{\beta}\psi'(\alpha))(\frac{\alpha}{\beta} - x)\} \\
&= (x - \frac{\alpha}{\beta}).
\end{aligned}$$

Note that $X - E_\theta(X) \in [\dot{l}_\theta] = \dot{\mathcal{P}}$; in fact, $X - E_\theta(X) = -\dot{l}_\beta(X)$.
Similarly, $\tilde{l}_B(x) = \dot{q}_B(\theta)I^{-1}(\theta)\dot{l}_\theta(x)$; unfortunately, this does not simplify much, largely due to the fact that $1_{[0,x_0]}(X) - F_\theta(x_0) \notin [\dot{l}_\theta] = \dot{\mathcal{P}}$.
(iii) The information bound for estimation of q_A is

$$\begin{aligned}
I^{-1}(P|q_A, \mathcal{P}) &= \dot{q}_A^T I^{-1}(\theta)\dot{q}_A \\
&= \frac{1}{\beta}(1, -\frac{\alpha}{\beta}) \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1} \begin{pmatrix} 1 \\ -\alpha/\beta \end{pmatrix} \frac{1}{\beta} \\
&= \frac{\alpha}{\beta^2} = Var_\theta(X).
\end{aligned}$$

Similarly,

$$I^{-1}(P|q_B, \mathcal{P}) = = \dot{q}_B^T I^{-1}(\theta)\dot{q}_B,$$

which does not simplify appreciably because $1_{[0,x_0]}(X) - F_\theta(x_0) \notin [\dot{l}_\theta] = \dot{\mathcal{P}}$. However, since we know that $\tilde{l}_B = \Pi(1_{[0,x_0]}(x) - F(x_0)|\dot{\mathcal{P}})$, it follows easily that

$$I^{-1}(P|q_B, \mathcal{P}) < E_\theta(1_{[0,x_0]}(X) - F_\theta(x_0))^2 = F_\theta(x_0)(1 - F_\theta(x_0));$$

i.e. it is possible to improve on the natural nonparametric estimator $\mathbb{F}_n(x_0)$ of $q_B(\theta) = F_\theta(x_0)$ when the model holds.