Statistics 581, Problem Set 2, Solutions
Wellner; 10/8/2014

1. (a) If $W \sim \chi^2_2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W, distribution function F_W, and inverse distribution function F_W^{-1} explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi^2_2$ and $\Theta \sim \text{Uniform}(-\pi/2, \pi/2)$. [Note that $g(\theta) = \tan(\theta)$ is a periodic function of θ with period π: $g(\theta) = g(\theta + \pi)$ for all θ. Thus the inverse function $g^{-1}(\theta) = \arctan(\theta)$ is usually taken to be the inverse of g restricted to $\theta \in (-\pi/2, \pi/2)$ where $\tan(\theta)$ is strictly increasing, negative for $-\pi/2 < \theta < 0$ and positive for $0 < \theta < \pi/2$.]
(c) Use the results of (a) and (b) to show (using Theorem 2.3.1, Chapter 2 notes, page 13) how to use two independent Uniform$(0, 1)$ random variables U and V to generate two standard normal random variables.

Solution: (a) If $W \sim \chi^2_2 = \text{Gamma}(1, 1/2)$, the density function is given by $f_W(w) = (1/2)e^{-w/2}1_{[0,\infty)}(w)$; i.e. $W \sim \text{Exponential}(1/2)$. Hence the distribution function is $F_W(w) = 1 - \exp(-w/2)$ for $w \ge 0$, and the inverse distribution function is $F_W^{-1}(u) = -2\log(1 - u)$.
(b) The joint density of (X, Y) is given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi}\exp(-(x^2 + y^2)/2) \qquad \text{for}(x, y) \in R^2 .$$

Moreover, $x = r\cos(\theta)$ and $y = r\sin(\theta)$ uniquely for $r \in (0, \infty)$ and $\theta \in (-\pi, \pi]$. However, the inverse function $(x, y) \mapsto (\sqrt{x^2 + y^2}, \arctan(y/x)) \equiv (r, \theta)$ maps both the half planes $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $\{(x, y) \in \mathbb{R}^2 : x < 0\}$ into $\{(r, \theta) : r > 0, -\pi/2 < \theta < \pi/2)\}$. Hence both of these half planes contribute separately to the joint density of (r, θ). Since the Jacobian of the transformation $(r, \theta) \mapsto (x(r, \theta), y(r, \theta))$ is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \left| \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \right| = r\cos^2(\theta) + r\sin^2(\theta) = r ,$$

we find that the joint density of (R, Θ) is given by

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= f_{X,Y}(r\cos(\theta), r\sin(\theta))1_{(0,\infty)}(r\cos(\theta)) \\ &\quad + f_{X,Y}(r\cos(\theta + \pi), r\sin(\theta + \pi))1_{(-\infty,0)}(r\cos(\theta + \pi)) \\ &= \frac{1}{2\pi}\exp(-r^2/2)r + \frac{1}{2\pi}\exp(-r^2/2)r \\ &= \frac{1}{\pi}\exp(-r^2/2)r \qquad \text{on} \ \ (0, \infty) \times (-\pi/2, \pi/2] \\ &= r\exp(-r^2/2) \cdot \frac{1}{\pi} = f_R(r)f_\Theta(\theta) . \end{aligned}$$

Thus R and Θ are independent with densities $f_R(r) = r\exp(-r^2/2)1_{(0,\infty)}(r)$ and $f_\Theta(\theta) = \pi^{-1}1_{(-\pi/2,\pi/2]}(\theta)$. Note that the distribution function of R is given by

$$F_R(r) = \int_0^r f_R(y)dy = \int_0^r y\exp(-y^2/2)dy = 1 - \exp(-r^2/2)\,.$$

It follows easily from this that

$$F_{R^2}(x) = P(R^2 \le x) = P(R \le \sqrt{x}) = 1 - \exp(-x/2)$$

for $x \in [0,\infty)$; i.e. $R^2 \sim$ Exponential$(1/2) =$ Gamma$(1, 1/2) = \chi^2_2$.
(c) If U and V are independent Uniform$(0,1)$ random variables, we can use the inverse transformation to first obtain

$$R^2 \equiv F^{-1}_{\chi^2_2}(U) = -2\log(1-U) \sim \chi^2_2 \qquad \text{and} \qquad \Theta \equiv 2\pi V \sim \text{ Uniform}(0, 2\pi)$$

note that R^2 and Θ are independent by independence of U and V. Then in view of (b)

$$(X, Y) \equiv (R\cos(\Theta), R\sin(\Theta)) \sim N_2(0, I)\,.$$

2. Suppose that $X_1, X_2, \ldots$ are iid Exponential(λ). Let $M_n \equiv \min_{1\le i\le n} X_i$ and $T_n \equiv \max_{1\le i\le n} X_i$.
(a) Show that $nM_n \stackrel{d}{=}$ exponential(λ).
(b) Show that $T_n - (1/\lambda)\log n \to_d (1/\lambda)T$ where T has the double exponential extreme value distribution function given by $P(T \le x) = \exp(-\exp(-x))$.
(c) Now suppose that $X_1, \ldots, X_n$ are iid with distribution function F satisfying $0 < F'(0) < \infty$; here $F'(0)$ is the right-derivative of F at 0:

$$\lim_{x\searrow 0} \frac{F(x) - F(0)}{x} = F'(0).$$

Show that $nM_n \to_d$ exponential $(F'(0))$.

Solution: (a) Now

$$P(nM_n > x) \;\; = \;\; P\left(\min_{1\le k\le n} X_k > \frac{x}{n}\right)$$

$$
\begin{aligned}
&= P\left(X_1 > \frac{x}{n}, \ldots, X_n > \frac{x}{n}\right) \\
&= P\left(X_1 > \frac{x}{n}\right) \cdots P\left(X_n > \frac{x}{n}\right) \\
&= P(X_1 > x/n)^n = \{\exp(-\lambda x/n)\}^n \\
&= \exp(-\lambda x).
\end{aligned}
$$

Hence $nM_n \stackrel{d}{=}$ exponential(λ).
(b) For the maximum T_n,

$$
\begin{aligned}
P(T_n - (1/\lambda)\log n \le x) &= P(\max_{1\le i\le n} X_i \le x + (1/\lambda)\log n) \\
&= P(X_1 \le x + (1/\lambda)\log n)^n \\
&= (1 - \exp(-\lambda x - \log n))^n \\
&= \left(1 - \frac{\exp(-\lambda x)}{n}\right)^n \to \exp(-\exp(-\lambda x)) \\
&= P(T \le \lambda x)
\end{aligned}
$$

where T has the extreme value distribution function $P(T \le x) = \exp(-\exp(-x)$ for $x \in \mathbb{R}$.
(c) In this case, the reasoning is much the same in part (a) at the beginning:

$$
\begin{aligned}
P(nM_n > x) &= P(X_1 > x/n)^n \\
&= \left(1 - \frac{nF(x/n)}{n}\right)^n \qquad (1)
\end{aligned}
$$

Since F has a derivative F' (from the right) at 0 and $F(0) = 0$ we have

$$
nF(x/n) = \frac{F(x/n) - F(0)}{x/n} \cdot x \to F'(0)x \qquad \text{as} \quad n \to \infty.
$$

Thus the expression on the right side of (1) converges to $\exp(-F'(0)x)$ as $n \to \infty$; i.e. $nM_n \to$ exponential $(F'(0))$.

3. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$
Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}\,;
$$

i.e.

$$E(Y-EY)^2 = E\{E[(Y-E(Y|X))^2|X]\} + E\{[E(Y|X)-E(Y)]^2\}\,.$$

(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi^2_n(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).
(d) Show that

$$\frac{\chi^2_n(\delta)-(n+\delta)}{\sqrt{2n+4\delta}} \to_d N(0,1)$$

as either $n \to \infty$ or $\delta \to \infty$.

Solution: (a) We compute directly:

$$\begin{aligned} Var(Y) &= E[Y-E(Y)]^2 = E[Y-E(Y|X)+E(Y|X)-E(Y)]^2 \\ &= E[Y-E(Y|X)]^2 + 2E[(Y-E(Y|X)][E(Y|X)-E(Y)] \\ &\qquad + E[E(Y|X)-E(Y)]^2 \\ &= E\{E\{[Y-E(Y|X)]^2|X\}\} + 0 + Var[E(Y|X)] \\ &= E\{Var[Y|X]\} + Var[E(Y|X)] \end{aligned}$$

since, by computing conditionally,

$$\begin{aligned} &E[(Y-E(Y|X)][E(Y|X)-E(Y)] \\ &\quad = E\{E\{[(Y-E(Y|X)][E(Y|X)-E(Y)]|X\}\} \\ &\quad = E\{[E(Y|X)-E(Y)]E\{[Y-E(Y|X)]|X\}\} \\ &\quad = E\{[E(Y|X)-E(Y)]\{E(Y|X)-E(Y|X)\}\} \\ &\quad = E\{[E(Y|X)-E(Y)]\cdot 0\} \\ &\quad = 0\,. \end{aligned}$$

(b) A geometric interpretation of (a) is that $Y-E(Y|X)$ is orthogonal to $E(Y|X)-E(Y)$ in $L_2(\Omega,\mathcal{A},P) = L_2(P)$, thus the identity in (a) can be intepreted as a "pythagorean theorem". Also note that $Y-E(Y|X)$ is orthogonal to any function $g(X)$: much as in the last part of (a)

$$\begin{aligned} &E[(Y-E(Y|X)]g(X) \\ &\quad = E\{E\{[(Y-E(Y|X)]g(X)|X\}\} \\ &\quad = E\{g(X)E\{[Y-E(Y|X)]|X\}\} \\ &\quad = E\{g(X)\{E(Y|X)-E(Y|X)\}\} \\ &\quad = E\{g(X)\cdot 0\} \\ &\quad = 0\,. \end{aligned}$$

(c) Now $(Y|K) \sim \chi^2_{2K+n}$ where $K \sim \text{Poisson}(\delta/2)$, so

$$E(Y) = E\{E(Y|K)\} = E\{2K+n\} = n + 2(\delta/2) = n + \delta\,.$$

Furthermore, using part (a) we get

$$\begin{aligned} Var(Y) &= E\{Var(Y|K)\} + Var\{E(Y|K)\} \\ &= E\{2(2K+n)\} + Var\{2K+n\} \\ &= 4(\delta/2) + 2n + 4(\delta/2) \\ &= 2n + 4\delta\,. \end{aligned}$$

(d) First suppose that $n \to \infty$. (d) First note that if $Y \sim \chi^2_n(\delta)$, then $Y =_d (Z_1 + \sqrt{\delta})^2 + Z_2^2 + \cdots + Z_n^2$ where $Z_1, Z_2, \ldots, Z_n$ are independent, $Z_1, \ldots, Z_n \sim N(0,1)$. We can write, with $T_i \equiv Z_i^2 - 1$ for $i = 2, \ldots, n$ (having mean 0 and variance 2),

$$\begin{aligned} \frac{\chi^2_n(\delta) - (n+\delta)}{\sqrt{2n+4\delta}} &=_d \frac{(Z_1 + \sqrt{\delta})^2 - (1+\delta)) + (Z_2^1 - 1) + \cdots + (Z_n^2 - 1)}{\sqrt{2n+4\delta}} \\ &= \frac{(Z_1^2 - 1) + \cdots + (Z_n^2 - 1)}{\sqrt{2n+4\delta}} + \frac{2\sqrt{\delta}Z_1}{\sqrt{2n+4\delta}} \\ &= \frac{T_1 + \cdots + T_n}{\sqrt{2n}} \frac{\sqrt{2n}}{\sqrt{2n+4\delta}} + \frac{2\sqrt{\delta}Z_1}{\sqrt{2n+4\delta}} \end{aligned}$$

where, as $n \to \infty$,

$$\frac{2\sqrt{\delta}Z_1}{\sqrt{2n+4\delta}} \to_p 0; \qquad \text{and} \qquad \frac{T_1 + \cdots + T_n}{\sqrt{2n}} \to_d N(0,1)$$

by the CLT. Hence by Slutsky's theorem

$$\frac{\chi^2_n(\delta) - (n+\delta)}{\sqrt{2n+4\delta}} \to_d N(0,1) \cdot 1 + 0 = N(0,1) \qquad \text{as} \;\; n \to \infty\,.$$

If n is fixed and $\delta \to \infty$,

$$\frac{T_1 + \cdots + T_n}{\sqrt{2n+4\delta}} \to_p 0$$

while

$$\frac{2\sqrt{\delta}Z_1}{\sqrt{2n+4\delta}} \to_d Z_1 \sim N(0,1)\,.$$

Hence the desired conclusion follows from Slutsky's theorem (Proposition 2.2.9).

4. Ferguson, ACILST, #4, page 6:
   Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.

   Solution: If $X_n = n$ with probability $p_n = 1/n^2$ and $X_n = 0$ otherwise, then $E|X_n| = np_n = 1/n$ while $E|X_n|^2 = n^2 p_n = 1$ for all n. More generally, if $X_n = a_n$ with $a_n \to \infty$ and $p_n = 1/a_n^\gamma$ with $\gamma > 1$, then $E(X_n) = 1/a_n^{\gamma-1} \to 0$ while $E(X_n^2) = a_n^2/a_n^\gamma = a_n^{2-\gamma} \to \infty$ if $1 < \gamma < 2$.

5. Ferguson, ACILT, #6, page 12. (This is related to Proposition 3.1 on page 12 of Chapter 0 and to Propositions 1.13 and 1.14 on pages 9 and 10 of Chapter 2.)

   Solution: (a) Suppose that $f_n(x) \to g(x)$ for all x where f_n and g are densities with respect to some dominating measure μ. Then, with $h_n \equiv g - f_n$, $\int (h_n^+ - h_n^-) d\mu = \int h_n d\mu = \int (g - f_n) d\mu = 1 - 1 = 0$, so that $\int h_n^+ d\mu = \int h_n^- d\mu$. Hence it follows that

$$\int |f_n - g| d\mu = \int |g - f_n| d\mu = \int h_n^+ d\mu + \int h_n^- d\mu = 2 \int h_n^+ d\mu$$

   where $h_n^+(x) = (g(x) - f_n(x))^+ \to 0$ for every x and $h_n^+ \le g$ with $\int g d\mu = 1 < \infty$. It follows by the dominated convergence theorem that

$$\int |f_n - g| d\mu = 2 \int h_n^+ d\mu \to 0.$$

   (b) See Proposition 3.1, chapter 0 notes.

6. **Optional Bonus Problem 1:**
   For $\theta > 0$, $\theta \ne 1$, let

$$C_\theta(u,v) \equiv \log\{1 + (\theta^u - 1)(\theta^v - 1)/(\theta - 1)\}/\log\theta.$$

(i) Find the density $c_\theta(u,v)$ corresponding to the distribution function $C_\theta(u,v)$.
(ii) Show that C_θ is a distribution function on $[0,1]^2$ with uniform marginal distributions.
(iii) Show that $C_\theta(u,v) \to u \cdot v$ for $0 < u, v < 1$ as $\theta \to 1$.
(iv) Show that if $(U,V) \sim C_\theta$ then $(1-U, 1-V) \stackrel{d}{=} (U,V)$.

Solution: (i) By straightforward differentiation,

$$c_\theta(u,v) = \frac{\partial^2}{\partial u \partial v} C_\theta(u,v) \quad = \quad \frac{\theta^{u+v}(\log\theta/(\theta-1))}{\left(1 + \frac{(\theta^u-1)(\theta^v-1)}{\theta-1)}\right)^2}.$$

It is easily seen that $c_\theta(u,v) \geq 0$ for all $\theta \in (0,1) \cup (1,\infty)$, and $C_\theta(u,v) = \int_0^u \int_0^v c_\theta(r,s) dr ds$.
(ii) C_θ is a distribution function by the last line of (i) with $c_\theta \geq 0$ together with $C_\theta(1,1) = 1$.
C_θ has uniform marginal distributions since $C_\theta(u,1) = \log\{1 + (\theta^v - 1)\}/\log\theta = v$ for $0 < v < 1$ and $C_\theta(1,v) = \log\{1 + (\theta^u - 1)\}/\log\theta = u$, $0 < u < 1$.
(iii) By two applications of L'Hopital's rule,

$$\lim_{\theta \to 1} C_\theta(u,v) = u \cdot v \quad \text{for} \quad 0 < u < 1, \quad 0 < v < 1.$$

(iv) To see that $(1-U, 1-V) \stackrel{d}{=} (U,V)$ we calculate

$$\begin{aligned}
&P_\theta(1-U \leq u, 1-V \leq v) \\
&\quad = \quad P_\theta(U \geq 1-u, V \geq 1-v) \\
&\quad = \quad 1 - C_\theta(1-u,1) - C_\theta(1,1-v) + C_\theta(1-u,1-v) \\
&\quad = \quad 1 - (1-u) - (1-v) + C_\theta(1-u,1-v) \\
&\quad = \quad u + v - 1 + C_\theta(1-u,1-v) \\
&\quad = \quad C_\theta(u,v)
\end{aligned}$$

since

$$\begin{aligned}
&C_\theta(1-u,1-v) - C_\theta(u,v) \\
&\quad = \frac{1}{\log\theta}\left\{\log\left\{1 + \frac{(\theta^{1-u}-1)(\theta^{1-v}-1)}{\theta-1}\right\} - \log\left\{1 + \frac{(\theta^u-1)(\theta^v-1)}{\theta-1}\right\}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\log\theta}\log\left\{\frac{1+\frac{(\theta^{1-u}-1)(\theta^{1-v}-1)}{\theta-1}}{1+\frac{(\theta^{u}-1)(\theta^{v}-1)}{\theta-1}}\right\} \\
&= \frac{1}{\log\theta}\log\left\{\frac{\theta+\theta^{2-u-v}-\theta^{1-u}-\theta^{1-v}}{\theta+\theta^{u+v}-\theta^{u}-\theta^{v}}\right\} \\
&= \frac{1}{\log\theta}\log\left\{\frac{\theta^{1-u-v}\{\theta^{u+v}+\theta-\theta^{v}-\theta^{u}\}}{\theta^{u+v}+\theta-\theta^{u}-\theta^{v}}\right\} \\
&= 1-u-v.
\end{aligned}
$$