**Statistics 581, Problem Set 1 Solutions**

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1. Ferguson, ACILST, #2, page 6.

   **Solution:** If $X_n \sim$ Uniformly on $\{1/n, 2/n, \ldots, 1\}$ then

   $$F_n(t) \;=\; P(X_n \le t) = \frac{\#\text{of } j/n \le t}{n} = \frac{\lfloor nt \rfloor}{n}.$$

   Note that $|F_n(t) - t| \le 1/n$ for each fixed $0 \le t \le 1$. Then $F_n(t) \to F(t) = t$ for all $0 \le t \le 1$. That is, $X_n \to_d X \sim \text{Uniform}(0,1)$. These $X_n$'s do not necessarily converge in probability to $X$ because the random variables $X_n$ are not necessarily defined on the same probability space.

2. (Continuation of the previous problem). Now suppose that $U \sim \text{Uniform}(0,1)$ and for each $n \ge 1$ define $V_n \equiv \sum_{j=1}^n (j/n) 1_{((j-1)/n, j/n]}(U)$.

   (a) Show that $V_n \stackrel{d}{=} X_n$ where $X_n$ is as in problem 2.
   (b) Show that $V_n \to_p U$.

   **Solution:** (a) Note that $P(V_n = j/n) = P(U \in ((j-1)/n, j/n]) = 1/n = P(X_n = j/n)$. Thus $V_n \stackrel{d}{=} X_n$.
   (b) To see that $V_n \to_p U$, note that

   $$P(|V_n - U| > \epsilon) = \begin{cases} n(1/n - \epsilon) = 1 - n\epsilon\,, & \text{if } 1/n > \epsilon \\ 0\,, & \text{if } 1/n \le \epsilon\,. \end{cases}$$

   Hence it follows that $V_n \to_p U$.

3. Suppose that $X_n$ has the Binomial distribution, Binomial$(n,p)$ for some $0 < p < 1$.
   (a) For fixed $k$, find $\lim_{n\to\infty} P(X_n \le k-1 | X_n \le k)$.
   (b) Now let the distribution of $Y_n$ be the conditional distribution of $X_n$ given $X_n \le k$ with $k \ge 1$ fixed. Express the result in (a) in the form $Y_n \to_d Y$.

**Solution:** (a) Since $X_n \sim \text{Binomial}(n,p)$ it follows that

$$\begin{aligned} P(X_n \le k-1|X_n \le k) &= \frac{P(X_n \le k-1)}{P(X_n \le k)} = \frac{\sum_{j=0}^{k-1} \binom{n}{j} p^j (1-p)^{n-j}}{\sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j}} \\ &= 1 - \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k} + \sum_{j=0}^{k-1} \binom{n}{j} p^j (1-p)^{n-j}} \\ &= 1 - \frac{1}{1 + \sum_{j=0}^{k-1} \frac{k!(n-k)!}{j!(n-j)!} p^{j-k} (1-p)^{k-j}} \\ &\to 1 - \frac{1}{1+0} = 0 \end{aligned}$$

since for $j \in \{0, \ldots, k-1\}$ we have $(n-k)!/(n-j)! \to 0$ as $n \to \infty$.
(b) The result of (a) implies that $P(X_n = k|X_n \le k) \to 1$ as $n \to \infty$ and that $P(X_n \le j|X_n \le k) \le P(X_n \le k-1|X_n \le k) \to 0$ for $j \in \{0, \ldots, k-1\}$. Thus $Y_n \to_d Y_0$ where $Y_0 = k$ with probability 1.

4. Suppose that $X$ has an inverse power distribution with distribution function $F_\alpha(x) = 1 - x^{-\alpha}$ for $x \in [1, \infty)$ where $\alpha > 0$.
(a) Show that $E(X) = \alpha/(\alpha - 1)$ for $\alpha > 1$ and $Var(X) = \alpha/((\alpha - 2)(\alpha - 1)^2)$ for $\alpha > 2$. (b) Let $Y_\alpha = (\alpha - 1)X - \alpha$ (so that $Y_\alpha$ has mean 0 and variance converging to 1 as $\alpha \to \infty$). Show that $Y_\alpha \to_d Y_\infty$ for some random variable $Y_\infty$ and find the distribution function of $Y_\infty$.

**Solution:** (a) By straightforward computation, $f_\alpha(x) = \alpha x^{-\alpha-1} 1_{[1,\infty)}(x)$ and hence for $\alpha > r$ we have

$$\begin{aligned} E(X^r) &= \int_1^\infty x^r f_\alpha(x) dx = \alpha \int_1^\infty x^{-(\alpha+1-r)} dx \\ &= \frac{\alpha}{\alpha - r}. \end{aligned}$$

This yields $E(X) = \alpha/(\alpha - 1)$ when $r = 1$ and $E(X^2) = \alpha/(\alpha - 2)$ when $r = 2$. Thus

$$Var(X) = E(X^2) - (E(X))^2 = \alpha \left\{ \frac{1}{\alpha - 2} - \frac{\alpha}{(\alpha - 1)^2} \right\} = \frac{\alpha}{(\alpha - 2)(\alpha - 1)^2}.$$

(b) It is easily seen that $Y_\alpha \equiv (\alpha - 1)X - \alpha$ has $E(Y_\alpha) = 0$ and $Var(Y_\alpha) = \alpha/(\alpha - 2) \to 1$ as $\alpha \to \infty$. Furthermore, for any fixed

$y \geq -1$,

$$\begin{aligned}
P(Y_\alpha \leq y) &= P((\alpha-1)X - \alpha \leq y) = P(X \leq (y+\alpha)/(\alpha-1)) \\
&= 1 - \left(\frac{\alpha+y}{\alpha-1}\right)^{-\alpha} = 1 - \left(1 + \frac{y+1}{\alpha-1}\right)^{-\alpha} \\
&\to 1 - \exp(-(y+1)), \quad \text{as} \quad \alpha \to \infty.
\end{aligned}$$

Thus $Y_\alpha \to_d Y_\infty \equiv Z - 1$ where $Z \sim$ exponential(1).

5. (a) Lehmann and Casella, TPE, problem 1.2, page 62.
   (b) Lehmann and Casella, TPE, problem 1.3, page 62.

   **Solution:** (i) First solution – via the Cauchy-Schwarz inequality: First recall the Cauchy-Schwarz inequality in $R^n$: if $u, v \in R^n$, then $(u'v)^2 \leq (u'u)(v'v)$ with equality iff $u = cv$ for some real number $c$. Now extend this as follows: if $\Sigma$ is positive definite and $x, y \in R^n$, then

$$(x'y)^2 = (\Sigma^{1/2}x)'(\Sigma^{-1/2}y) \leq (x'\Sigma x)(y'\Sigma^{-1}y)$$

with equality iff $\Sigma^{1/2}x = c\Sigma^{-1/2}y$; i.e. iff $x = c\Sigma^{-1}y$.

Now consider $X$, a random vector in $R^n$, with $E(X) = \mathbf{1}\theta$ and $Cov(X, X) = E[(X - E(X))(X - E(X))'] = \Sigma$, where $\mathbf{1} = (1, \ldots, 1)' \in R^n$. A linear estimator $\alpha'X = \alpha_1 X_1 + \ldots + \alpha_n X_n$ is unbiased for $\theta$ iff $\theta = E(\alpha'X) = \alpha'E(X) = (\alpha'\mathbf{1})\theta$ for all $\theta$; i.e., iff $\alpha'\mathbf{1} = 1$. The variance of $\alpha'X$ is $Var(\alpha'X) = \alpha'\Sigma\alpha$. To find the best such estimator, we must find

$$\min\{\alpha'\Sigma\alpha : \alpha'\mathbf{1} = 1\}.$$

But by the Cauchy-Schwarz inequality, if $\alpha'\mathbf{1} = 1$, then

$$\alpha'\Sigma\alpha \geq 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$$

with equality iff $\alpha = c\Sigma^{-1}\mathbf{1}$. The condition $\alpha'\mathbf{1} = \mathbf{1}$ then implies that $c = 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$, so the optimal $\alpha$ is $\alpha_0 \equiv \Sigma^{-1}\mathbf{1}/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$, and the optimal linear unbiased estmator is $\alpha_0'X = (\mathbf{1}'\Sigma^{-1}X)/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$ whose variance is $Var(\alpha_0'X) = 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$.
The solutions to 1.2, 1.3(a), and 1.3(b) now follow:
1.2: In this case $\Sigma = \sigma^2 I$, so $\alpha_0 = \mathbf{1}(1/\sigma^2)/(\mathbf{1}'I\mathbf{1}/\sigma^2) = \mathbf{1}(1/n)$.
1.3(a): The inverse of the matrix diag$(1/c_i)$ is just diag$(c_i)$. This

implies that $\alpha_0' X = (\sum_1^n a_i X_i)/(\sum_1^n c_i)$ and $Var(\alpha_0' X) = \sigma^2/\sum c_i$.
1.3(b): The inverse of the matrix with 1 on the diagonal and $\rho$ off the diagonal is of the form $a$ in the diagonal entries and $b$ in the off-diagonal entries for some $a, b$. Hence $\Sigma^{-1}\mathbf{1} = \sigma^{-2}(a + (n-1)b)\mathbf{1}$, which leads to $\mathbf{1}'\Sigma^{-1}X = \sigma^{-2}(a+(n-1)b)(X_1 + \ldots + X_n)$, and $\mathbf{1}'\Sigma^{-1}\mathbf{1} = \sigma^{-2}(a + (n-1)b)n$. Hence we find that $\alpha_0' X = \sum_1^n X_i/n$. But $\Sigma\mathbf{1} = \sigma^2(1 + (n-1)\rho)\mathbf{1}$, so $\mathbf{1} = \sigma^2(1+(n-1)\rho)\Sigma^{-1}\mathbf{1} = (1+(n-1)\rho)(a+(n-1)b)\mathbf{1}$, and hence $[a + (n-1)b] = [1 + (n-1)\rho]^{-1}$. Therefore

$$Var(\alpha_0' X) \quad = \quad \frac{\sigma^2}{n}[1 + (n-1)\rho] \begin{cases} > \sigma^2/n & \text{if } \rho > 0 \\ < \sigma^2/n & \text{if } -1/(n-1) \le \rho < 0 \end{cases} .$$

[Note that if $\rho < -1/(n-1)$, the matrix $\Sigma$ of this form *is not a covariance matrix*!

6. **Bonus problem 1:** Let $\mathcal{X} = (0,1)$, $\mathcal{Y} = (0,1)$, both equipped with the Borel sets and Lebesgue measure. Let

$$g(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad \text{for } (x,y) \in (0,1) \times (0,1) .$$

Show that:
(a) $\int_0^1 (\int_0^1 g(x,y)dy)dx = \pi/4$.
(b) $\int_0^1 (\int_0^1 g(x,y)dx)dy = -\pi/4$.
(c) Why does Fubini's theorem fail here?
**Solution:** (a) It is easily seen that

$$\int_0^1 g(x,y)dy = \frac{y}{x^2 + y^2}\Big|_{y=0}^1 = \left(\frac{1}{1+x^2} - 0\right) = \frac{1}{1+x^2} ,$$

and hence

$$\int_0^1 (\int_0^1 g(x,y)dy)dx = \int_0^1 \frac{1}{1+x^2}dx = \arctan(x)|_0^1 = \pi/4 .$$

(b) Similarly,

$$\int_0^1 g(x,y)dx = -\frac{x}{x^2 + y^2}\Big|_{x=0}^1 = -\left(\frac{1}{1+y^2} - 0\right) = \frac{-1}{1+y^2} ,$$

and hence

$$\int_0^1 \left(\int_0^1 g(x,y)dy\right)dx = -\int_0^1 \frac{1}{1+y^2}dy = -\arctan(y)|_0^1 = -\pi/4\,.$$

(c) It is clear that the hypothesis $g \in L_1(\lambda \times \lambda)$ fails in this example; if it held, the two iterated integrals in (a) and (b) must be the same by the Fubini part of the Fubini-Tonelli theorem. To see that $g \notin L_1(\mu \times \nu)$ directly, note that the function $g$ is non-negative on the set $0 \le y \le x \le 1$, and non-positive on the set $0 \le x \le y \le 1$. Thus

$$\int_0^1 \int_0^1 |g(x,y)|dxdy = \int\int_{\{0\le y\le x\le 1\}} g(x,y)dxdy + \int\int_{\{0\le x\le y\le 1\}} -g(x,y)dxdy\,,$$

where, by the Tonelli part of the Fubini-Tonelli theorem

$$\begin{aligned}
\int\int_{\{0\le y\le x\le 1\}} g(x,y)dxdy &= \int_0^1 \left(\int_y^1 g(x,y)dx\right) dy \\
&= \int_0^1 \left(\frac{-x}{x^2+y^2}\Big|_{x=y}^1\right) dy \\
&= \int_0^1 \left(\frac{-1}{1+y^2} - \frac{-y}{2y^2}\right) dy \\
&= \int_0^1 \left(\frac{1}{2y} - \frac{1}{1+y^2}\right) dy \\
&= \infty - \pi/4 = \infty\,.
\end{aligned}$$

By symmetry it follows that

$$\int\int_{\{0\le x\le y\le 1\}} -g(x,y)dxdy = \infty\,,$$

and hence we have $\int\int |g(x,y)|dxdy = \infty$. Thus $g \notin L_1(\lambda \times \lambda)$ on $[0,1]\times[0,1]$.

7. **Bonus problem 2:** Suppose that $X \sim F$ on $R^+ \equiv [0,\infty)$, $Y \sim G$ on $R^+$, and $X$ and $Y$ are independent random variables. Let $Z = \min\{X,Y\} = X \wedge Y$ and $\Delta = 1\{X \le Y\}$. (This is *right-censored data*: if we view $X$ as a survival time, and $Y$ as a censoring time, then $Z = X$ when $X \le Y$, but $Z = Y$ when $X > Y$.)

(a) Find the joint distribution of $(Z, \Delta)$.
(b) If $X \sim$ Exponential$(\lambda)$ and $Y \sim$ Exponential$(\mu)$, show that $Z$ and $\Delta$ are independent.
[Hint: for (a), compute $P(Z \le z, \Delta = 1)$ and $P(Z \le z, \Delta = 0)$.]
**Solution:** (a) Since $Z = \min\{X, Y\} = X \wedge Y$ and $\Delta = 1\{X \le Y\}$, it follows that

$$H_{uc}(z) \equiv P(X \le z, X \le Y) = \int_{[0,z]} (1 - G(x-))dF(x)\,,$$

and

$$H_c(z) \equiv P(Y \le z, X > Y) = \int_{[0,z]} (1 - F(y))dG(y)\,.$$

These two sub-distribution functions completely determine the joint distribution function $H$ of $(Z, \Delta)$ since

$$P(Z \le z, \Delta \le \delta) = \begin{cases} 0, & \text{if } \delta < 0\,, \\ H_c(z), & \text{if } 0 \le \delta < 1\,, \\ H_c(z) + H_{uc}(z), & \text{if } 1 \le \delta < \infty\,. \end{cases}$$

Note that

$$1 - H_c(z) - H_{uc} = P(Z > z) = (1 - F(z))(1 - G(z))\,,$$

so the marginal d.f. of $Z$ is

$$H(z, 1) = H_c(z) + H_{uc}(z) = 1 - (1 - F(z))(1 - G(z))\,.$$

(b) When $1 - F(x) = \exp(-\lambda x)$ and $1 - G(x) = \exp(-\mu x)$, then

$$1 - H(z, 1) = (1 - F(z))(1 - G(z)) = \exp(-(\lambda + \mu)x)\,,$$

while

$$P(\Delta = 1) = P(X \le Y) = H_{uc}(\infty) = \frac{\lambda}{\lambda + \mu}\,,$$

so $Z \sim$ Exponential$(\lambda + \mu)$, $\Delta \sim$ Bernoulli$(\lambda/(\lambda + \mu))$. Furthermore,

$$H_{uc}(z) = \int_0^z e^{-\mu x} \lambda e^{-\lambda x}\, dx = \frac{\lambda}{\lambda + \mu}\,(1 - \exp(-(\lambda + \mu)z))$$

$$H_c(z) = \int_0^z e^{-\lambda x} \lambda e^{-\mu x}\, dx = \frac{\mu}{\lambda + \mu}\,(1 - \exp(-(\lambda + \mu)z))\;,$$

so that $Z$ and $\Delta$ are independent in this case.