**Statistics 581, Problem Set 9**

Wellner; 11/19/2014

**Reading:** Chapter 4, Sections 1-2;
Ferguson, ACLST, Chapters 18- 20, pages 119-125, 133-139; Chapter 22, pages 144-150;
Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.
**Due:** Wednesday, November 26, 2014.

1. Consider the Weibull family of example 3.2.5 and problem set #6, problem 1: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities

$$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}\exp(-(\frac{x}{\alpha})^\beta)1_{[0,\infty)}(x)$$

where $\theta \equiv (\alpha, \beta) \in (0,\infty) \times (0,\infty) \subset R^2$. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with density function $p_\theta$.
(a) If $X \sim P_\theta \in \mathcal{P}$, show that the distributions of $\log X$ form a location and scale family from a Gumbel (extreme value) density on $R$. (This amounts to a rephrasing of the statement of problem 1 in problem set 6.)
(b) Use the result of (a) to construct method of moments estimators or quantile based estimators $\overline{\theta}_n$ of $\theta = (\alpha, \beta)$.
(c) Show that the method of moments or quantile estimators $\overline{\theta}_n$ of $\theta$ are asymptotically normal, and find the asymptotic distribution; i.e. show that

$$\sqrt{n}(\overline{\theta}_n - \theta) \to_d N_2(0, \Sigma) \qquad \text{for some} \qquad \Sigma.$$

[We will use these estimators as "starting points" approximate (or one-step) maximum likelihood estimators in the next problem .]

2. (Problem #1, continued).
(a) Does a maximum likelihood estimate of $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ exist? Is it unique? (See Lehmann and Casella, Example 6.1, page 468.)
(b) Compute an approximate (one - step) maximum likelihood estimate $\check{\theta}$ of $\theta$ using the method of moment (or quantile) estimators $\overline{\theta}_n$ as the preliminary estimators based on the following data (with $n = 12$):

$$1,\ 1,\ 2,\ 3,\ 12,\ 21,\ 46,\ 54,\ 65,\ 109,\ 317,\ 413.$$

[These are failure times in seconds for "breakdown" of an insulating fluid between two electrodes subject to a voltage of 40 kV. – from Nelson, *Applied Life Data Analysis*, page 252, modified slightly.]
(c) Compute the maximum likelihood estimator $\hat{\theta}_n$, and compare it with the one step estimator computed in (b).

3. (a) Ferguson, ACILST, problem 17.2, page 117: I would suggest modifying Ferguson's definition of the density to:

$$p_\theta(x) \equiv f(x|\theta) = 2\left\{ \frac{x}{\theta} 1_{[0,\theta]}(x) + \frac{1-x}{1-\theta} 1_{(\theta,1]}(x) \right\}.$$

(b) Do our hypotheses A0-A2 hold in this example?
(c) Compute $K(P_{\theta_0}, P_\theta)$ where $P_\theta$ has density as given in this problem.
(d) Do our hypotheses A3 and A4 hold in this example? Why or why not?
(e) Does there exist an estimator $\overline{\theta}_n$ of $\theta$ which is $n^{1/2}-$consistent?
In connection with this problem note that the likelihood function $L(\theta)$ given on Ferguson's page 215 is *not correct*: the formula there should be replaced by

$$L_n(\theta|\underline{X}) \;=\; \left(\frac{2}{\theta}\right)^k \prod_{j=1}^{k} X_{(j)} \cdot \left(\frac{2}{1-\theta}\right)^{n-k} \prod_{j=k+1}^{n} (1 - X_{(j)}) \;\text{ if }\; X_{(k)} \le \theta < X_{(k+1)}.$$

4. (a) Lehmann and Casella, problem 6.3.1, page 501.
(b) Lehmann and Casella, problem 6.3.2, page 501.
(c) Lehmann and Casella, problem 6.3.4, page 501.

5. **Optional bonus problem 1:** Suppose that $X_1, \ldots, X_n$ are i.i.d. log-normal$(\mu, \sigma^2)$ with density

$$p_\theta(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left( -\frac{(\log x - \mu)^2}{2\sigma^2} \right) 1_{(0,\infty)}(x).$$

Here $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^+ \equiv (-\infty, \infty) \times (0, \infty)$.
(a) Find the MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$ of $\theta = (\mu, \sigma^2)$.
(b) Show that $\log X \overset{d}{=} \mu + \sigma Z \sim N(\mu, \sigma^2)$ where $Z \sim N(0,1)$.
(c) Suppose that $\nu(P_\theta) = q(\theta) = E_\theta(X)$. Express $q(\theta)$ explicitly as a function of $\theta$.
(d) Suggest a natural nonparametric estimator $\overline{\nu}_n$ of $E_\theta(X)$.
(e) Find the asymptotic variance of $\sqrt{n}(\overline{\nu}_n - \nu(P_\theta))$ for the estimator $\overline{\nu}_n$ you proposed in (d).
(f) What is the MLE $\hat{\nu}_n$ of $\nu = \nu(P_\theta)$ assuming that the log-normal model is true? What do our results in chapter 3 say about the asymptotic distribution of $\sqrt{n}(\hat{\nu}_n - \nu(P_\theta))$ (assuming that the model holds)?
(g) Compare the variances you found in (e) and (f). Which estimator do you prefer if the log-normal model holds?

6. **Optional bonus problem 2:** Lehmann and Casella, TPE, problem 6.3.22, page 503, reworded as follows. (In other words, prove (vi) of theorem 1.2, pages 5-6, chapter 4 notes). Suppose that $X_1, \ldots, X_n$ are i.i.d. with density $p_\theta$, $\theta \in \Theta \subset R^k$, satisfying the hypotheses of theorem 4.1, page 463 (the Cramér conditions given in (A) - (D) on pages 462-463). Show that the following Local Asymptotic Normality (LAN) result holds for the (local) log- likelihood ratios: with

$$L_n(\theta) \equiv \log(\prod_{i=1}^{n} p_\theta(X_i)) = \sum_{i=1}^{n} \log p_\theta(X_i),$$

for a fixed $\theta_0 \in \Theta$,

$$
\begin{aligned}
L_n(\theta_0 + n^{-1/2}\underline{t}) - L_n(\theta_0) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \underline{t}^T \dot{\underline{l}}_\theta(X_i) - \frac{1}{2}\underline{t}^T I(\theta_0)\underline{t} + o_p(1) \\
&\to_d N(0, t^T I(\theta_0) t) - \frac{1}{2} t^T I(\theta_0) t \\
&\stackrel{d}{=} N(-\sigma^2/2, \sigma^2)
\end{aligned}
$$

under $P_{\theta_0}$ where $\sigma^2 \equiv \underline{t}^T I(\theta_0)\underline{t}$. (The convergence in the last display actually holds under the considerably weaker hypothesis of Hellinger differentiability of $p_\theta$ at $\theta_0$, as stated in Corollary 3 of section 3.3, page 28, of the Chapter 3 notes.)

7. **Optional bonus problem 3:** Suppose that $(Y|Z) \sim \text{Poisson}(\lambda e^{\gamma Z})$, and $Z \sim Bernoulli(\eta)$, and $\theta = (\lambda, \gamma, \eta)$. Let $X = (Y, Z)$, and suppose that we observe $X_1, \ldots, X_n$ i.i.d. as $X$.
(a) Find the score equations for estimation of $\theta$.
(b) Give conditions on the data $X_1, \ldots, X_n = (Y_1, Z_1), \ldots, (Y_n, Z_n)$ guaranteeing that the score equations have a unique solution which maximizes the likelihood. Call the resulting estimators $\widehat{\theta}_n = (\widehat{\lambda}_n, \widehat{\gamma}_n, \widehat{\eta}_n)$.
(c) What does theorem 4.1.2 (Chapter 4, page 5), say about the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta_0)$ when the distribution of the data is given by $P_{\theta_0}$?
(d) Suppose that $\theta_1 \neq \theta_0$ is the "true" value of the parameter $\theta$, and we consider the likelihood ratio $L_n(\theta_1)/L_n(\theta_0)$ where $L_n(\theta) \equiv \prod_{i=1}^n p_\theta(X_i)$. Show that $n^{-1}\log(L_n(\theta_1)/L_n(\theta_0) \to_p$ some constant, and identify the constant explicitly in terms of $\theta_1$, $\theta_0$.