

Statistics 581, Problem Set 8

Wellner; 11/12/2014

Reading: Chapter 3, Sections 3-5; start reading Chapter 4 (to be handed out on Monday, November 17);

Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;

Lehmann and Casella, Chapter 6, especially sections 6.1-6.2, pages 429-443.

Due: Wednesday, November 19, 2014.

1. (a) Show that if $\theta_n = cn^{-1/2}$ and T_n is the Hodges super-efficient estimator discussed in class, then the sequence $\{\sqrt{n}(T_n - \theta_n)\}$ is uniformly square-integrable.
 (b) Lehmann and Casella, Problem 2.13, page 501.
 (c) Let $R_n(\theta) \equiv nE_\theta(T_n - \theta)^2$ where T_n is the Hodges superefficient estimator as in Example 3.3.1 (so $T_n = \delta_n$ of Example 2.5, Lehmann and Casella pages 440 - 443). Show that $R_n(n^{-1/4}) \rightarrow \infty$ as $n \rightarrow \infty$.
2. Suppose that $Z \sim N(0, 1)$ and, for $\mu \in R$ and $\sigma > 0$, that $X = \mu + \sigma Z \sim P_{\mu, \sigma} = N(\mu, \sigma^2)$.
 (a) Compute the likelihood ratio

$$\frac{dP_{\mu, \sigma}}{dP_{0, \sigma}}(x) = \frac{\sigma^{-1}\phi((x - \mu)/\sigma)}{\sigma^{-1}\phi(x/\sigma)} \quad \text{and} \quad Y \equiv \log \frac{dP_{\mu, \sigma}}{dP_{0, \sigma}}(X).$$

What is the distribution of Y under $P_{0, \sigma}$ and under $P_{\mu, \sigma}$?

- (b) Plot the function $l(\mu; X) \equiv \log(dP_{\mu, \sigma}/dP_{0, \sigma})(X)$ as a function of μ .
 - (c) Find the maximum value of the function $l(\mu; X)$ in B (as a function of μ) and the value of $\mu \equiv \hat{\mu}$ which achieves the maximum.
 - (d) What is the distribution of $\hat{\mu}$ under $P_{0, \sigma}$ and under $P_{\mu, \sigma}$? What is the distribution of $l(\hat{\mu}; X)$ under $P_{0, \sigma}$ and under $P_{\mu, \sigma}$?
3. Suppose that $(T|Z) \sim \text{Weibull}(\lambda^{-1}e^{-\gamma Z}, \beta)$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ . Thus the conditional cumulative hazard function $\Lambda(t|z)$ is given by

$$\Lambda_{\gamma, \lambda, \beta}(t|z) = (\lambda e^{\gamma Z} t)^\beta = \lambda^\beta e^{\beta \gamma Z} t^\beta$$

and hence

$$\lambda_{\gamma, \lambda, \beta}(t|z) = \lambda^\beta e^{\beta \gamma Z} \beta t^{\beta-1}.$$

(Recall that $\lambda(t) = f(t)/(1 - F(t))$ and

$$\Lambda(t) \equiv \int_0^t \lambda(s) ds = \int_0^t (1 - F(s))^{-1} dF(s) = -\log(1 - F(t))$$

if F is continuous.) Thus it makes sense to re-parametrize by defining $\theta_1 \equiv \beta \gamma$ (this is the parameter of interest since it reflects the effect of the covariate Z), $\theta_2 \equiv \lambda^\beta$, and $\theta_3 \equiv \beta$. This yields

$$\lambda_\theta(t|z) = \theta_3 \theta_2 \exp(\theta_1 z) t^{\theta_3-1}$$

You may assume that

$$a(z) \equiv (\partial/\partial\eta) \log g_\eta(z)$$

exists and $E\{a^2(Z)\} < \infty$. Thus Z is a “covariate” or “predictor variable”, θ_1 is a “regression parameter” which affects the intensity of the (conditionally) Weibull variable T , and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ where $\theta_4 \equiv \eta$.

- (a) Derive the joint density $p_\theta(t, z)$ of (T, Z) for the re-parametrized model.
- (b) Find the information matrix for θ . What does the structure of this matrix say about the effect of $\eta = \theta_4$ being known or unknown about the estimation of $\theta_1, \theta_2, \theta_3$?
- (c) Find the information and information bound for θ_1 if the parameters θ_2 and θ_3 are known.
- (d) What is the information bound for θ_1 if just θ_3 is known to be equal to 1?
- (e) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_3 is known.
- (f) Find the information $I_{11 \cdot (2,3)}$ and information bound for θ_1 if the parameters θ_2 and θ_3 are unknown. (Here both θ_2 and θ_3 are in “the second block”.)
- (g) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_2 and θ_3 are unknown.
- (h) Specialize the calculations in (d) - (g) to the case when $Z \sim \text{Bernoulli}(\theta_4)$ and compare the information bounds.

4. **Optional bonus problem 1:** Suppose that $X \sim F_\theta = \text{exponential}(\theta)$ with density $f_\theta(x) = \theta e^{-\theta x} 1_{(0, \infty)}(x)$ and $Y \sim G_\eta$ independent of X with densities $\{g_\eta : \eta \in \mathbb{R}^+\}$, a regular parametric model on $(0, \infty)$. Consider the following three scenarios for observation of X or functions of X :

- (a) Uncensored: we observe X and Y .
- (b) Right-censored: we observe $T(X, Y) = (X \wedge Y, 1\{X \leq Y\}) \equiv (\min\{X, Y\}, 1\{X \leq Y\}) \equiv (Z, \Delta)$.
- (c) Interval-censored (case 1): we observe $S(X, Y) = (Y, 1\{X \leq Y\}) \equiv (Y, \Delta)$.

In each of the three scenarios (a), (b), (c):

- (i) Find the joint density of (X, Y) and joint distributions of $T(X, Y)$ and $S(X, Y)$.
- (ii) Find the scores for θ and η . (Let $(\partial/\partial\eta) \log g_\eta(y) \equiv a(y)$ with $a \in L_2^0(G_\eta)$.)
- (iii) Compute and compare $I_{X,Y}(\theta)$, $I_{T(X,Y)}(\theta)$, and $I_{S(X,Y)}(\theta)$. Make the comparisons in general and then explicitly by making one or more choices of the family $\{g_\eta\}$.

5. **Optional bonus problem 2:** This is a continuation of problem 4 on the midterm exam.

- (a) Suppose that f is differentiable at x . What further assumptions on f' or f'' and b_n do you need to show that $\sqrt{nb_n} \{E(\hat{f}_n(x)) - f(x)\} \rightarrow 0$? (This says that the bias of \hat{f}_n for estimating x is $o((nb_n)^{-1/2})$.)
- (b) Combine the conclusion of (a) with the conclusion of problem 4(d) on the exam to find the limiting distribution of $\sqrt{2nb_n}(\hat{f}_n(x) - f(x))$.
- (c) Now suppose that $x, y \in \mathbb{R}$ satisfy $x \neq y$, $f(x) > 0$ and $f(y) > 0$. Find the

limiting joint distribution of

$$\sqrt{2nb_n} \begin{pmatrix} \hat{f}_n(x) - f(x) \\ \hat{f}_n(y) - f(y) \end{pmatrix}$$

under appropriate further hypotheses on derivatives of f at x and y and the sequence b_n .

(d) Compare the result in (c) with the joint limiting distribution of

$$\sqrt{n} \begin{pmatrix} \mathbb{F}_n(x) - F(x) \\ \mathbb{F}_n(y) - F(y) \end{pmatrix}$$

obtained in Chapter 2 (what is it explicitly?).

6. **Optional bonus problem 3:** Suppose that X_1, \dots, X_n are i.i.d. F on \mathbb{R} , and let \mathbb{F}_n denote the empirical d.f. of the X_i 's. Let Φ denote the standard normal distribution function, $\Phi(x) = \int_{-\infty}^x \phi(y) dy$ where $\phi(y) = (2\pi)^{-1/2} \exp(-y^2/2)$ is the standard normal density. Let $0 < a < 1$ and define a new estimator \tilde{F}_n of F by

$$\tilde{F}_n(x) = \begin{cases} (1-a)\Phi(x) + a\mathbb{F}_n(x), & \text{if } \|\mathbb{F}_n - \Phi\|_\infty \leq n^{-1/4}, \\ \mathbb{F}_n(x), & \text{if } \|\mathbb{F}_n - \Phi\|_\infty > n^{-1/4}. \end{cases}$$

- (a) Find the limiting distribution of the process $\{\sqrt{n}(\tilde{F}_n(x) - F(x)) : x \in \mathbb{R}\}$ when $F = \Phi$.
- (b) Find the limiting distribution of the process $\{\sqrt{n}(\tilde{F}_n(x) - F(x)) : x \in \mathbb{R}\}$ when $F \neq \Phi$.
- (c) Show that \tilde{F}_n is not a regular estimator of F at $F = \Phi$ (in an appropriate sense to be defined), but that F is a regular estimator of F at any $F \neq \Phi$.