Statistics 581, Problem Set 7

Wellner; 11/5/2014

Reading: Chapter 3, Sections 2-4;
Ferguson, ACILST, Chapters 19-20, pages 126-139;
Lehmann and Casella, pages 113-129, and 437-443.
Due: Wednesday, November 12, 2014.

1. Consider the two parameter location-scale model

$$\mathcal{P} = \left\{ P_\theta : \ \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta \right\}$$

where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right),$$

and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ.
(a) Calculate the information matrix $I(\theta)$ for θ.
(b) For which of the densities in (a)-(e) of problem set #6, problem 3, is $I_{12}(\theta)$ not zero?

2. Lehmann and Casella, TPE, Problem 6.6, page 142.

3. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \le m < k$. Let $\dot{\mathbf{l}} = \dot{\mathbf{l}}_\theta = (\dot{\mathbf{l}}_1, \dot{\mathbf{l}}_2)$ be the corresponding partition of the (vector of) scores $\dot{\mathbf{l}}$, and, with $\widetilde{\mathbf{l}} \equiv I^{-1}(\theta)\dot{\mathbf{l}}$, the *efficient influence function* for θ, let $\widetilde{\mathbf{l}} = (\widetilde{\mathbf{l}}_1, \widetilde{\mathbf{l}}_2)$ be the corresponding partition of $\widetilde{\mathbf{l}}$. In both cases, $\dot{\mathbf{l}}_1$, $\widetilde{\mathbf{l}}_1$ are $m-$vectors of functions, and $\dot{\mathbf{l}}_2$, $\widetilde{\mathbf{l}}_2$ are $k-m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where I_{11} is $m \times m$, I_{12} is $m \times (k-m)$, I_{21} is $(k-m) \times m$, I_{22} is $(k-m) \times (k-m)$. Also write $I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}$.
(a) Verify that:
$I^{11} = I_{11\cdot 2}^{-1}$ where $I_{11\cdot 2} \equiv I_{11} - I_{12} I_{22}^{-1} I_{21}$, $I^{22} = I_{22\cdot 1}^{-1}$ where $I_{22\cdot 1} \equiv I_{22} - I_{21} I_{11}^{-1} I_{12}$, $I^{12} = -I_{11\cdot 2}^{-1} I_{12} I_{22}^{-1}$, and $I^{21} = -I_{22\cdot 1}^{-1} I_{21} I_{11}^{-1}$.
This amounts to formulas (4) and (5) of section 3.2, page 19.
(b) Verify that
$\widetilde{\mathbf{l}}_1 = I^{11}\dot{\mathbf{l}}_1 + I^{12}\dot{\mathbf{l}}_2 = I_{11\cdot 2}^{-1}(\dot{\mathbf{l}}_1 - I_{12} I_{22}^{-1}\dot{\mathbf{l}}_2)$, and
$\widetilde{\mathbf{l}}_2 = I^{21}\dot{l}_1 + I^{22}\dot{\mathbf{l}}_2 = I_{22\cdot 1}^{-1}(\dot{\mathbf{l}}_2 - I_{21} I_{11}^{-1}\dot{\mathbf{l}}_1)$.
(c) Verify that $\widetilde{\mathbf{l}}_1 = I_{11}^{-1}\dot{\mathbf{l}}_1 - I_{11}^{-1} I_{12}\widetilde{\mathbf{l}}_2$ and hence that $I_{11\cdot 2}^{-1} = I_{11}^{-1} + I_{11}^{-1} I_{12} I_{22\cdot 1}^{-1} I_{21} I_{11}^{-1}$.
This amounts to (15) and (16) of section 3.2, page 21.

4. **Optional bonus problem:** Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty) \equiv \Theta\,.$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0; here $F_\theta(x) \equiv P_\theta(X \le x)$.
(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page 127, Table 6.1
(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
(iii) Find the efficient influence functions for estimation of q_A and q_B.
(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators $\overline{X}_n$ and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.