Statistics 581, Problem Set 3

Wellner; 10/8/2014

Reading: Course Notes: Chapter 2, pages 1 - 25; start Ferguson ACILST pages 26-60. Lehmann & Casella, TPE, pages 54-61 and pages 75-78.
Due: Wednesday, October 15, 2014.

1. Ferguson, ACILST, page 34, problem 1(a) (modified slightly)
   Suppose that $X_1, X_2, \ldots$ are i.i.d. in R^2 with distribution giving probability θ_1 to $(1,0)'$, probability θ_2 to $(0,1)'$, θ_3 to $(0,0)'$ and θ_4 to $(-1,-1)'$ where $\theta_j \ge 0$ for $j = 1,2,3,4$ and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1$.
   (a) Find $\mu = E(X_1)$.
   (b) Compute $E(X_1 X_1^T)$ and $\Sigma = E(X_1 - \mu)(X_1 - \mu)^T$.
   (c) Find the limiting distribution of $\sqrt{n}(\overline{X}_n - \mu)$ and describe the resulting approximation to the distribution of $\overline{X}_n$.
   (d) Find values of $(\theta_1, \ldots, \theta_4)$ such that Σ has rank 1 and $\det(\Sigma) = 0$.

2. Suppose that $X_1, X_2, \ldots$ are i.i.d. (μ, σ^2) with $\mu_4 < \infty$. Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $S_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ be the sample mean and sample variance respectively.
   (a) Show that
   $$\sqrt{n}\begin{pmatrix} \overline{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} \to_d \underline{Z} \sim N_2(0, \Sigma)$$
   where
   $$\begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}.$$
   (b) Suppose $\sigma > 0$. Use (a) to find the limiting distribution of the sample *signal-noise ratio* $D_n \equiv \overline{X}_n / S_n$; i.e. show that $\sqrt{n}(D_n - d) \to_d N(0, V^2)$ with $d \equiv \mu/\sigma$ and find V^2.

3. Ferguson, ACILST, page 34, problem 1(b) (modified slightly)
   Suppose that $X_1, \ldots, X_n$ is a sample from the Poisson distribution with parameter $\lambda > 0$: $P(X_1 = k) = \exp(-\lambda)\lambda^k/k!$, $k = 0, 1, \ldots$. Let $Z_n = (1/n)\sum_{i=1}^n 1_{[X_i = 0]}$.
   (a) What is the joint asymptotic distribution of
   $$\sqrt{n}((\overline{X}_n, Z_n)' - (\lambda, e^{-\lambda})')?$$
   (b) Let $p_0(\lambda) \equiv P_\lambda(X_1 = 0)$. What is the asymptotic distribution of $\hat{p}_0 \equiv p_0(\hat{\lambda}_n)$ where $\hat{\lambda}_n = \overline{X}_n$?
   (c) What is the joint asymptotic distribution of $(Z_n, \hat{p}_0)$ (after centering and rescaling)?
   (d) Compute the ratio of the asymptotic variances of the two estimators Z_n and $\hat{p}_0$ of $p_0(\lambda)$. Which estimator would you prefer if the Poisson model (assumption) holds? Which estimator would you prefer if the Poisson model (assumption) fails?

4. A sequence of random variables Y_n is *bounded in probability* and we write $Y_n = O_p(1)$ if

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} P(|Y_n| > \lambda) = 0;$$

i.e. for each $\epsilon > 0$ there exist λ_ϵ and N_ϵ such that $P(|Y_n| > \lambda_\epsilon) < \epsilon$ for all $n > N_\epsilon$.
(a) Show that if $Y_n \to_d Y$ for some random variable Y, then Y_n is bounded in probability. (This is Lehmann and Casella, problem 8.24, page 77.)
(b) Give an example of a sequence of random variables Y_n that is bounded in probability, but does not converge in distribution.
(c) Lehmann and Casella, problem 8.25, page 77.

5. Suppose that X is a random variable with finite fourth moment; $E|X|^4 < \infty$. Then $\mu_4 = E(X-\mu)^4$ is the fourth central moment of X. The ratio $\mu_4/\sigma^4 \equiv \kappa$ is the *kurtosis* of X (or of the distribution function F of X), and $\gamma_2 \equiv \mu_4/\sigma^4 - 3$ is called the *excess of kurtosis*; note that for any $N(\mu, \sigma^2)$ random variable, $\gamma_2 = 0$. Investigate the value of γ_2 for various classical distributions (t_r, uniform, bernoulli, Poission(λ), ...). How big can γ_2 be? How small can γ_2 be?

6. **Optional Bonus Problem 1.** Suppose that $X_1, X_2, \ldots$ are i.i.d. positive random variables, and define $\overline{X}_n \equiv n^{-1}\sum_{i=1}^n X_i$, $H_n \equiv 1/(n^{-1}\sum_{i=1}^n (1/X_i))$, and $G_n \equiv \{\prod_{i=1}^n X_i\}^{1/n}$ to be the *arithmetic, harmonic, and geometric* means respectively. We know that $\overline{X}_n \to_{a.s.} E(X_1) = \mu$ if and only if $E|X_1| < \infty$.
(a) Use the SLLN together with appropriate additional hypotheses to show that $H_n \to_{a.s.} 1/\{E(1/X_1)\} \equiv h$, and $G_n \to_{a.s.} \exp(E\{\log X_1\}) \equiv g$.
(c) Use the multivariate CLT and the delta method to find the joint limiting distribution of $\sqrt{n}(\overline{X}_n - \mu, H_n - h, G_n - g)$. You will need to impose or assume additional moment conditions to be able to prove this. Specify these additional assumptions carefully.

7. **Optional bonus problem 2:** (i) Ferguson, ACILST, page 34, problem 3.
(ii) Ferguson, ACILST, page 34, problem 4.