Statistics 581, Problem Set 2
Wellner; 10/1/2014

Reading: Course Notes: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 8-25.
Due: Wednesday, October 8, 2010.

1. (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W, distribution function F_W, and inverse distribution function F_W^{-1} explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(0, 2\pi)$.
(c) Use the results of (a) and (b) to show (using Theorem 2.3.4) how to use two independent $\text{Uniform}(0, 1)$ random variables U and V to generate two standard normal random variables.

2. Suppose that $X_1, X_2, \ldots$ are iid Exponential(λ). Let $M_n \equiv \min_{1 \le i \le n} X_i$ and $T_n \equiv \max_{1 \le i \le n} X_i$.
(a) Show that $nM_n \stackrel{d}{=} \text{exponential}(\lambda)$.
(b) Show that $T_n - (1/\lambda)\log n \to_d (1/\lambda)T$ where T has the double exponential extreme value distribution function given by $P(T \le x) = \exp(-\exp(-x))$.
(c) Now suppose that $X_1, \ldots, X_n$ are iid with distribution function F satisfying $0 < F'(0) < \infty$; here $F'(0)$ is the right-derivative of F at 0:
$$\lim_{x \searrow 0} \frac{F(x) - F(0)}{x} = F'(0).$$
Show that $nM_n \to_d$ exponential $(F'(0))$.

3. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that
$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\};$$
i.e.
$$E(Y - EY)^2 = E\{E[(Y - E(Y|X))^2|X]\} + E\{[E(Y|X) - E(Y)]^2\}.$$
(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).
(d) Show that
$$\frac{\chi_n^2(\delta) - (n + \delta)}{\sqrt{2n + 4\delta}} \to_d N(0, 1)$$
as either $n \to \infty$ or $\delta \to \infty$.

4. Ferguson, ACILST, #4, page 6:
   Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.

5. Ferguson, ACILT, #6, page 12. (This is related to Proposition 3.1 on page 12 of Chapter 0 and to Propositions 1.13 and 1.14 on pages 9 and 10 of Chapter 2.)

6. **Optional Bonus Problem 1:**
   For $\theta > 0$, $\theta \neq 1$, let

$$C_\theta(u, v) \equiv \log\{1 + (\theta^u - 1)(\theta^v - 1)/(\theta - 1)\}/\log\theta.$$

   (i) Find the density $c_\theta(u, v)$ corresponding to the distribution function $C_\theta(u, v)$.
   (ii) Show that C_θ is a distribution function on $[0, 1]^2$ with uniform marginal distributions.
   (iii) Show that $C_\theta(u, v) \to u \cdot v$ for $0 < u, v < 1$ as $\theta \to 1$.
   (iv) Show that if $(U, V) \sim C_\theta$ then $(1 - U, 1 - V) \stackrel{d}{=} (U, V)$.