

Statistics 581, Problem Set 2, corrected version

Wellner; 10/1/2014, corrected 10/2/14

Reading: Course Notes: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 8-25.

Due: Wednesday, October 8, 2010.

1. (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W , distribution function F_W , and inverse distribution function F_W^{-1} explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(-\pi/2, \pi/2)$. [Note that $g(\theta) = \tan(\theta)$ is a periodic function of θ with period π : $g(\theta) = g(\theta + \pi)$ for all θ . Thus the inverse function $g^{-1}(\theta) = \arctan(\theta)$ is usually taken to be the inverse of g restricted to $\theta \in (-\pi/2, \pi/2)$ where $\tan(\theta)$ is strictly increasing, negative for $-\pi/2 < \theta < 0$ and positive for $0 < \theta < \pi/2$.]
(c) Use the results of (a) and (b) to show (using Theorem 2.4.1, Chapter 2 notes, page 23) how to use two independent $\text{Uniform}(0, 1)$ random variables U and V to generate two standard normal random variables.
2. Suppose that X_1, X_2, \dots are iid $\text{Exponential}(\lambda)$. Let $M_n \equiv \min_{1 \leq i \leq n} X_i$ and $T_n \equiv \max_{1 \leq i \leq n} X_i$.
(a) Show that $nM_n \stackrel{d}{=} \text{exponential}(\lambda)$.
(b) Show that $T_n - (1/\lambda) \log n \rightarrow_d (1/\lambda)T$ where T has the double exponential extreme value distribution function given by $P(T \leq x) = \exp(-\exp(-x))$.
(c) Now suppose that X_1, \dots, X_n are iid with distribution function F satisfying $0 < F'(0) < \infty$; here $F'(0)$ is the right-derivative of F at 0:

$$\lim_{x \searrow 0} \frac{F(x) - F(0)}{x} = F'(0).$$

Show that $nM_n \rightarrow_d \text{exponential}(F'(0))$.

3. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$\text{Var}(Y) = E\{\text{Var}(Y|X)\} + \text{Var}\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{E[(Y - E(Y|X))^2|X]\} + E\{[E(Y|X) - E(Y)]^2\}.$$

- (b) Interpret (a) geometrically.
- (c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $\text{Var}(Y)$.

Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

(d) Show that

$$\frac{\chi_n^2(\delta) - (n + \delta)}{\sqrt{2n + 4\delta}} \rightarrow_d N(0, 1)$$

as either $n \rightarrow \infty$ or $\delta \rightarrow \infty$.

4. Ferguson, ACILST, #4, page 6:
Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow 1$.
5. Ferguson, ACILT, #6, page 12. (This is related to Proposition 3.1 on page 12 of Chapter 0 and to Propositions 1.13 and 1.14 on pages 9 and 10 of Chapter 2.)
6. **Optional Bonus Problem 1:**
For $\theta > 0$, $\theta \neq 1$, let

$$C_\theta(u, v) \equiv \log\{1 + (\theta^u - 1)(\theta^v - 1)/(\theta - 1)\} / \log \theta.$$

- (i) Find the density $c_\theta(u, v)$ corresponding to the distribution function $C_\theta(u, v)$.
- (ii) Show that C_θ is a distribution function on $[0, 1]^2$ with uniform marginal distributions.
- (iii) Show that $C_\theta(u, v) \rightarrow u \cdot v$ for $0 < u, v < 1$ as $\theta \rightarrow 1$.
- (iv) Show that if $(U, V) \sim C_\theta$ then $(1 - U, 1 - V) \stackrel{d}{=} (U, V)$.