STATISTICS 581:
Day 1 Quiz Solutions, Fall, 2010

1. **State** the (classical) Central Limit Theorem.

Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$, then

$$\sqrt{n}(\overline{X}_n - \mu) \to_d Y \sim N(0, \sigma^2).$$

That is,

$$P(\sqrt{n}(\overline{X}_n - \mu) \le x) \to P(Y \le x) = P(\sigma Z \le x) = \Phi(x/\sigma)$$

where $Z \sim N(0,1)$ and $P(Z \le z) \equiv \Phi(z) = \int_{-\infty}^{z} (2\pi)^{-1/2} \exp(-t^2/2)\, dt$

2. **State** the weak law of large numbers.

Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E|X_1| < \infty$, then

$$\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i \to_p \mu \equiv E(X_1).$$

3. Suppose that X has density $f(x) = 2x^{-3} 1_{[1,\infty)}(x)$.
(a) For what values of $r \in R$ is it true that $E(X^r) < \infty$?
(b) For the values of r for which the expectation is finite, compute it explicitly.
(c) If $X_1, X_2, \ldots$ are i.i.d. with the same density as X, does the law of large numbers hold?
(d) If $X_1, X_2, \ldots$ are i.i.d. with the same density as X, does the central limit theorem hold?

Solution: (a) We compute

$$\begin{aligned} E(X^r) &= \int_1^\infty x^r f(x) dx = \int_1^\infty 2x^{r-3} dx = \frac{2}{r-2} x^{r-2}|_1^\infty \\ &= \frac{2}{2-r} \quad \text{if } r < 2. \end{aligned}$$

Thus $E(X^r) < \infty$ if $r < 2$.
(b) The computation in (a) gave $E(X^r) = 2/(2-r)$.
(c) The WLLN holds since $E|X| = E(X) = 2 < \infty$.
(d) The central limit theorem fails since $E(X^2) = \infty$.

4. **Define** what is meant by:
(a) X_n converges in distribution to X for random variables X and X_n, $n \geq 1$.
(b) X_n converges in probability to X.

Solution: (a) $X_n \to_d X$ if

$$F_{X_n}(x) \equiv P(X_n \leq x) \to P(X \leq x) \equiv F_X(x)$$

for all $x \in C_{F_X}$, the set of continuity points of F_X.
(b) $X_n \to_p X$ if

$$P(|X_n - X| > \epsilon) \to 0 \quad \text{as} \quad n \to \infty$$

for every $\epsilon > 0$.

5. Suppose that U is a random variable with a Uniform$(0,1)$ distribution. For each integer $n \geq 2$ define $X_n = n 1_{[0,1/n]}(U)$.
(a) Does $X_n \to_d X$? (If so, identify the distribution of the limiting variable X.)
(b) Does $X_n \to_p X$? (If so, identify the limit variable X.)
(c) Compute $E(X_n)$. Does it converge to $E(X)$ for some X?

Solution: (a) & (b) $X_n \to_p 0 \equiv X$. To see this, let $\epsilon > 0$. Then

$$P(|X_n| > \epsilon) \leq P(U \leq 1/n) = 1/n \to 0.$$

Since $X_n \to_p 0$, we also have $X_n \to_d 0$.
(c)

$$\begin{aligned} E(X_n) &= E\{n 1_{[0,1/n]}(U)\} = n E 1_{[0,1/n]}(U) \\ &= n P(U \leq 1/n) = n(1/n) = 1. \end{aligned}$$

Thus $E(X_n) \to 1 \neq 0 = E(X)$ where $P(X = 0) = 1$.

6. Suppose that $X_1, \ldots, X_n, ...$ are independent and identically distributed Exponential(θ) random variables (i.e. $P(X > x) = \exp(-\theta x)$ for $x \geq 0$). Let $T_n = X_1 + \ldots + X_n$.
(a) What is the distribution of T_n?
(b) Does $\overline{X}_n = n^{-1} T_n \to_p$ something? If so, what is "something"?
(c) Does $\sqrt{n}(\overline{X}_n - p) \to_d$ something? If so, what is "something"?
(d) What is the Cramér-Rao bound for unbiased estimators of p?

Solution: (a) $T_n \sim$ Gamma(n, θ).
(b) First, for any $r > 0$

$$E(X^r) = \int_0^\infty x^r \theta \exp(-\theta x) dx = \theta^{-r} \int_0^\infty t^r e^{-t} dt = \theta^{-r} \Gamma(r+1).$$

Thus for $r = 1$ we have $E(X) = \theta^{-1}\Gamma(2) = \theta^{-1}$, and for $r = 2$ we have $E(X^2) = \theta^{-2}\Gamma(3) = 2\theta^{-2}$. Hence it follows that

$$Var(X) = E(X^2) - E(X)^2 = 2\theta^{-2} - (\theta^{-1})^2 = \theta^{-2}.$$

(c) From the WLLN $\overline{X}_n \to_p E(X_1) = \theta^{-1}$.
(d) From the CLT,

$$\sqrt{n}(\overline{X}_n - \theta^{-1}) \to_d N(0, \theta^{-2}).$$

(e) $p(x;\theta) = \theta \exp(-\theta x) 1_{(0,\infty)}(x)$, so

$$\log p(x;\theta) = \log \theta - \theta x,$$

and

$$\dot{\mathbf{l}}_\theta(x) = \theta^{-1} - x, \qquad \ddot{\mathbf{l}}_{\theta,\theta}(x) = -\theta^{-2}.$$

Thus the information for θ (in a sample of size $= 1$) is $I(\theta) = \theta^{-2}$, and the information lower bound for unbiased estimators of θ is $1/(nI(\theta)) = \theta^2/n$.