Statistics 581, Problem Set 2 Solutions
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1. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}.$$

(b) Interpret (a) geometrically.
(c) Suppose that $(Y|X) \sim N(-X^2/2, \theta X^2)$ where $X \sim$ Exponential(1) and $\theta > 0$. Compute $E(Y)$ and $Var(Y)$. Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

Solution: (a) We compute directly:

$$\begin{aligned} Var(Y) &= E[Y - E(Y)]^2 = E[Y - E(Y|X) + E(Y|X) - E(Y)]^2 \\ &= E[Y - E(Y|X)]^2 + 2E[(Y - E(Y|X)][E(Y|X) - E(Y)] \\ &\qquad + E[E(Y|X) - E(Y)]^2 \\ &= E\{E\{[Y - E(Y|X)]^2|X\}\} + 0 + Var[E(Y|X)] \\ &= E\{Var[Y|X]\} + Var[E(Y|X)] \end{aligned}$$

since, by computing conditionally,

$$\begin{aligned} &E[(Y - E(Y|X)][E(Y|X) - E(Y)] \\ &\quad = E\{E\{[(Y - E(Y|X)][E(Y|X) - E(Y)]|X\}\} \\ &\quad = E\{[E(Y|X) - E(Y)]E\{[Y - E(Y|X)]|X\}\} \\ &\quad = E\{[E(Y|X) - E(Y)]\{E(Y|X) - E(Y|X)\}\} \\ &\quad = E\{[E(Y|X) - E(Y)] \cdot 0\} \\ &\quad = 0\,. \end{aligned}$$

(b) A geometric interpretation of (a) is that $Y - E(Y|X)$ is orthogonal to $E(Y|X) - E(Y)$ in $L_2(\Omega, \mathcal{A}, P) = L_2(P)$, thus the identity in (a) can be intepreted as a "pythagorean theorem". Also note that $Y - E(Y|X)$ is orthogonal to any function $g(X)$: much as in the last part of (a)

$$\begin{aligned} &E\{[(Y - E(Y|X)]g(X)\} \\ &\quad = E\{E\{[(Y - E(Y|X)]g(X)|X\}\} \\ &\quad = E\{g(X)E\{[Y - E(Y|X)]|X\}\} \\ &\quad = E\{g(X)\{E(Y|X) - E(Y|X)\}\} \\ &\quad = E\{g(X) \cdot 0\} \\ &\quad = 0\,. \end{aligned}$$

(c) Now $(Y|X) \sim N(-X^2/2, \theta X^2)$ where $X \sim$ Exponential(1), so

$$E(Y) = E\{E(Y|X)\} = E\{-X^2/2\} = -\frac{1}{2}E(X^2) = -\frac{1}{2}2 = -1.$$

Furthermore, using part (a) we get

$$\begin{aligned} Var(Y) &= E\{Var(Y|X)\} + Var\{E(Y|X)\} \\ &= E\{\theta X^2\} + Var\{-X^2/2\} \\ &= \theta E(X^2) + \frac{1}{4}Var(X^2) \\ &= 2\theta + \frac{1}{4}\{E(X^4) - (E(X^2))^2\} \\ &= 2\theta + \frac{1}{4}\{24 - 2^2\} = 2\theta + 5 \end{aligned}$$

since

$$E(X^r) = \int_0^\infty x^r e^{-x} dx = \Gamma(r+1) = r!$$

for any integer r.

2. (a) The case $r = 1$ of Chebyshev's inequality is known as Markov's inequality and is usually written $P(|X| \geq \epsilon) \leq E(|X|)/\epsilon$ for an arbitrary random variable X and $\epsilon > 0$. For each fixed $\epsilon > 1$, find a distribution for X with $E(X) = 0$ and $E(|X|) = 1$ that gives equality in Markov's inequality.
(b) For an arbitrary random variable X, show that

$$P(|X| \geq \epsilon) \leq \frac{E\cosh(X) - 1}{\cosh(\epsilon) - 1}.$$

Solution: Suppose that $P(X = \pm\epsilon) = p/2$, $P(X = 0) = 1 - p$. Then $E(X) = 0$, $E|X| = \epsilon p = 1$ if $p = 1/\epsilon$, and for $\epsilon > 1$ we have

$$P(|X| \geq \epsilon) = p = 1/\epsilon = E|X|/\epsilon.$$

(b) Since $g(x)$ is symmetric (i.e. $g(-x) = g(x)$ for all x), $g(x) \geq 0$, and $g(x) \nearrow$ as $x \nearrow$, it follows that

$$\begin{aligned} g(\epsilon)P(|X| \geq \epsilon) &= g(\epsilon)E1_{\{|X|\geq\epsilon\}} = E\{g(\epsilon)1_{\{|X|\geq\epsilon\}}\} \\ &\leq E\{g(X)1_{\{|X|\geq\epsilon\}}\} \\ &\leq E\{g(X)\}. \end{aligned}$$

The stated conclusion follows by taking $g(x) = \cosh(x) - 1$ and rearranging. Note that this is a special case of the "Basic inequality" given in Proposition 1.9, chapter 2 (course notes), page 6.

3. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

Solution: See Ferguson, ACILST, page 63. Find Γ orthogonal so that $\Gamma'\Sigma\Gamma = D$ where D is diagonal. Now $\Gamma\Gamma' = I$, so if $\Sigma^2 = \Sigma$ we have $D^2 = \Gamma'\Sigma\Gamma\Gamma'\Sigma\Gamma = \Gamma'\Sigma^2\Gamma = \Gamma'\Sigma\Gamma = D$ and conversely. Moreover, since Σ is of rank k, D is of rank k, and this together with $D^2 = D$ implies that D has k 1's on the diagonal and $n - k$ 0's. Without loss, assume that Γ has been chosen so that the k ones occur in the the first r positions of the diagonal matrix D; thus

$$D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is $k \times k$. Moreover, note that

$$\begin{aligned} D\Gamma'\mu &= \Gamma'\Sigma\Gamma\Gamma'\mu = \Gamma'\Sigma\mu \\ &= \Gamma'\mu, \end{aligned}$$

and this implies that the last $n-k$ components of $\Gamma'\mu$ are all zero. Now let $Y = \Gamma'X$ (much as in the proof of theorem 1.3.2 of the notes). Then $Y \sim N_n(\Gamma'\mu, D)$, $Y'Y = X'\Gamma\Gamma'X = X'X$, and by (d) of page 16, section 1.3, $X'X = Y'Y \sim \chi_k^2(\delta)$ where $\delta = (\Gamma'\mu)'(\Gamma'\mu) = \mu'\Gamma\Gamma'\mu = \mu'\mu$.

4. Ferguson, ACILST, #2, page 6:
(a) Suppose that $X_n \sim \text{Uniform}\{1/n, 2/n, \ldots, n/n\}$. Show that $X_n \to_d X \sim \text{Uniform}(0,1)$. Does $X_n \to_p X$?
(b) Suppose that $Y_n = \sum_{k=1}^n (k/n) 1_{[(k-1)/n, k/n)}(U)$ where $U \sim \text{Uniform}[0,1]$. Show that $Y_n \sim \text{Uniform}\{1/n, 2/n, \ldots, n/n\}$, and $Y_n \to_p U$.

Solution: (a) For $0 \le x \le 1$,

$$P(X_n \le x) \;=\; \frac{1}{n}\sum_{i=1}^n 1_{[i/n,1]}(x) = \frac{1}{n}\sum_{i=1}^n 1_{[i/n \le x]} = [nx]/n \to x\,;$$

here $[x]$ = greatest integer less than or equal to x. Thus $X_n \to X \sim$ Uniform$(0,1)$. X_n does not necessarily converge in probability to X because all the different random variables involved could be defined on different probability spaces.
(b) Now $P(Y_n = k/n) = P(U \in [(k-1)/n, k/n)) = 1/n$ for $k = 1, \ldots, n$, so $Y_n \sim$ Uniform$\{1/n, \ldots, n/n\}$. Furthermore,

$$P(|Y_n - U| \ge \epsilon) = \begin{cases} n(1/n - \epsilon), & \text{if } 0 \le \epsilon \le 1/n \\ 0, & \text{if } \epsilon > 1/n\,, \end{cases}$$

and this clearly converges to 0 as $n \to \infty$.

5. (a) Lehmann and Casella, #3.5, page 64.
(b) Lehmann and Casella, #3.6, page 64.
(c) Lehmann and Casella, #3.7, page 64.

Solution: (a) (i) Suppose that S is not closed. Then there exists a sequence $\{x_n\} \subset S$ such that $x_n \to x_0 \in S^c$. But then, for every $\epsilon > 0$ there is an open ball $B(x_0, \epsilon)$ such that $x_n \in B(x_0, \epsilon)$ for $n \ge N_\epsilon$. Since each x_n is a support point, $P(B(x_0, \epsilon)) > 0$ for each $\epsilon > 0$. But for any open set A with $x_0 \in A$, $B(x_0, \epsilon) \subset A$ for some $\epsilon > 0$, and hence $P(A) \ge P(B(x_0, \epsilon) > 0$. But this implies $x_0 \in S$. Contradiction. Thus S is closed.
(ii) $P(S) = 1$. From (i) S is closed, so S^c is open. Since $x \in S^c$ if and only if $x \in A_x$ with A_x an open rectangle satisfying $P(A) = 0$. Thus $S^c \subset \cup_x A_x$. By the Lindelöf theorem, for any such open covering $\{A_x\}_{x \in S^c}$ of $S^c \subset R^d$, there is a countable subcollection $\{A_{x_n}\}$ which covers S^c: $S^c \subset \cup_n A_{x_n}$. Then we have

$$P(S^c) \le P(\cup_n A_{x_n}) \le \sum_n P(A_{x_n}) = \sum_n 0 = 0\,.$$

Hence $P(S) = 1$.
(iii) We want to show that $S = \cap\{C : \ C \text{ closed}, P(C) = 1\}$. From (i) and (ii) we know that S is in the collection of sets on the right side, so it follows that $S \supset \cap\{C : \ C \text{ closed}, P(C) = 1\}$. Thus it remains to show that $S \subset \cap\{C : \ C \text{ closed}, P(C) = 1\}$. Equivalently, it remains to show that $S^c \supset \cup\{C^c : C^c \text{ open}, \ P(C^c) = 0\}$. But if $x \in \cup\{C^c : C^c \text{ open}, \ P(C^c) = 0\}$, then $x \in C^c$ for some C^c open with

$P(C^c) = 0$, and hence also $x \in A \subset C^c$ for some open rectangle A (an open ball centered at x for the metric $\|y\| = \max_{1 \le i \le d} |x_i|$) with $P(A) \le P(C^c) = 0$. Hence $x \in S^c$.
(b) Suppose that P and Q are equivalent: i.e. $Q \prec\prec P$ and $P \prec\prec Q$. Then for any open set A, $P(A) = 0$ if and only if $Q(A) = 0$. This implies that for any closed set A^c,

$$P(A^c) = 1 \qquad \text{if and only if} \qquad Q(A^c) = 1\,.$$

This implies that the minimal closed set S_P with $P(S_P) = 1$ is also the minimal closed set S_Q with $Q(S_Q) = 1$; i.e. $S_P = \text{supp}(P) = \text{supp}(Q) = S_Q$.
(c) Since $P(X = 1/n) = p_n > 0$ for $n = 1, 2, \ldots$ with $\sum_1^\infty p_n = 1$, it follows that $\text{supp}(P) = \{0, \ldots, 1/n, \ldots, 1/2, 1\}$, which is closed. Similarly, Since $Q(X = 1/n) = q_n > 0$ for $n = 1, 2, \ldots$ with $\sum_1^\infty q_n = 1/2$, and $Q(X = 0) = 1/2$, it follows that $\text{supp}(Q) = \{0, \ldots, 1/n, \ldots, 1/2, 1\} = \text{supp}(P)$. But $P(\{0\}) = 0$ while $Q(\{0\}) = 1/2$, so $Q \prec\prec P$ fails. Thus Q and P are not equivalent.