

## Statistics 581, Problem Set 1 Solutions

Wellner; 10/5/2010

1. Let  $X$  and  $Y$  be i.i.d.  $\text{Uniform}(0, 1)$  random variables. Define  $U = X - Y$ ,  $V = \max(X, Y) = X \vee Y$ .

- (i) What is the range of  $(U, V)$ ?  
(ii) Find the joint density function  $f_{U,V}(u, v)$  of the pair  $(U, V)$ . Are  $U$  and  $V$  independent?

**Solution:** (i) The range of  $(X, Y)$  is

$A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ . The range of  $(U, V)$  is

$B = \{(u, v) : 0 \leq u \leq 1, u \leq v \leq 1\} \cup \{(u, v) : -1 \leq u < 0, -u \leq v \leq 1\}$ .

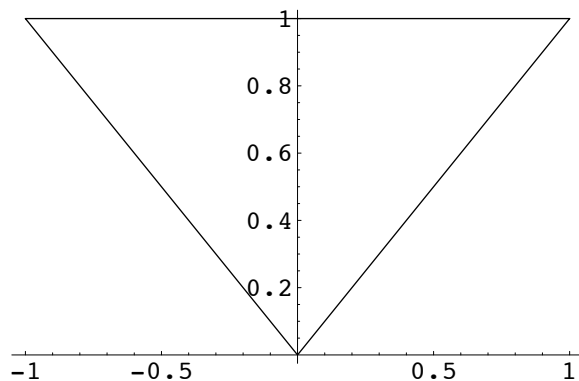


Figure 1: Range of  $U, V$ .

(ii) First solution - via Jacobians: The transformation  $(X, Y) \rightarrow (U, V)$  is 1-1 and onto from  $A$  to  $B$ . On the set  $x < y$ , its inverse is given by  $X = U + V, Y = V$ ; on the set  $x > y$ , its inverse is given by  $X = V, Y = V - U$ . These mappings are continuously differentiable on  $B^* \equiv B \setminus \{(u, v) : (0, v)\} = B \setminus$  a null set. On  $B^*$  the Jacobian of the transformations are

$$\det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \quad \text{if } x < y, \quad \det \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = 1 \quad \text{if } x > y. \quad (1)$$

Thus by the usual transformation of densities formula, the joint density of  $(U, V)$  is obtained from  $f_{X,Y}(x, y) = 1_{[0,1]}(x)1_{[0,1]}(y)$  as follows:

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(x(u, v), y(u, v)) \left| \det \frac{\partial(x, y)}{\partial(u, v)} \right| 1_{[x(u,v) < y(u,v)]} \\ &\quad + f_{X,Y}(x(u, v), y(u, v)) \left| \det \frac{\partial(x, y)}{\partial(u, v)} \right| 1_{[x(u,v) > y(u,v)]} \\ &= (1_{[0,1]}(u+v)1_{[0,1]}(v)1_{[u+v < v]} + 1_{[0,1]}(v)1_{[0,1]}(v-u)1_{[v > v-u]}) \\ &= 1_B(u, v). \end{aligned}$$

Thus the joint density of  $(U, V)$  is uniform on  $B$ . The random variables  $U$  and  $V$  are clearly *not* independent since the range of  $(U, V)$  is not a product set in  $R^2$ ; moreover, the joint density of  $(U, V)$  does not factor into the product of its marginal densities. [The marginal densities are given by

$$f_U(u) = \int f_{U,V}(u, v) dv = \begin{cases} \int_u^1 dv = 1 - u, & u \in [0, 1] \\ \int_{-u}^1 dv = 1 + u, & u \in [-1, 0) \end{cases}$$

and

$$f_V(v) = \int f_{U,V}(u, v) du = \int_{-v}^v du = 2v 1_{[0,1]}(v).$$

Second solution by direction calculation of the joint distribution function: Note that we can write

$$\begin{aligned} &P(U \leq u, V \leq v) \\ &= P(X - Y \leq u, X \vee Y \leq v) = P(X - Y \leq u, X \leq v, Y \leq v) \\ &= P(Y \geq X - u, X \leq v, Y \leq v) \\ &= \begin{cases} v^2 - \frac{1}{2}(v-u)^2, & \text{if } 0 \leq u \leq v \leq 1, \\ \frac{1}{2}(v+u)^2, & \text{if } -1 \leq u < 0, 0 < -u \leq v \leq 1. \end{cases} \end{aligned}$$

(This is easy by pictures!) Computing  $(\partial^2/\partial u\partial v)P(U \leq u, V \leq v)$  on each of these pieces separately again yields  $f_{U,V}(u, v) = 1_B(u, v)$ . Also note that the marginal distribution functions of  $U$  and  $V$  are given by  $F_U(u) = (1/2)(1+u)^2 1_{[-1,0)}(u) + \{1 - \frac{1}{2}(1-u)^2\} 1_{[0,1]}(u)$  on  $-1 \leq u \leq 1$  and  $F_V(v) = v^2$  for  $0 \leq v \leq 1$ .

2. Ferguson, ACILST, #6, page 7. (This is known as the Polya-Cantelli lemma; see Chapter 2, Proposition 2.11, page 10.)

**Solution.** See Ferguson, ACILST page 173.

3. Suppose that for  $\theta \in R$ ,

$$f_\theta(u, v) = \{1 + \theta(1 - 2u)(1 - 2v)\} 1_{[0,1]^2}(u, v).$$

- (a) For what values of  $\theta$  is  $f_\theta$  a density function on  $[0, 1]^2$ ?  
 (b) For the set of  $\theta$ 's you identified in (a), find the corresponding distribution function  $F_\theta$  and show that it has Uniform(0, 1) marginal distributions.  
 (c) If  $(U, V) \sim F_\theta$ , compute the correlation  $\rho(U, V) \equiv \rho$  as a function of  $\theta$ . Does this show any difficulty with this family of distributions as a model of dependence?

**Solution:** (a) For  $f_\theta$  to be a density function, we must have  $f_\theta(u, v) \geq 0$  for all  $(u, v) \in [0, 1]^2$  and

$$\int_0^1 \int_0^1 f_\theta(u, v) dudv = 1. \tag{2}$$

Now

$$\int_0^1 \int_0^1 f_\theta(u, v) dudv = 1 + \theta \int_0^1 \int_0^1 (1 - 2u)(1 - 2v) dudv = 1$$

for all  $\theta \in R$  since

$$\int_0^1 \int_0^1 (1 - 2u)(1 - 2v) dudv = \int_0^1 (1 - 2u) du \int_0^1 (1 - 2v) dv = 0 \cdot 0 = 0,$$

and hence (2) holds for all  $\theta$ . The requirement that  $f_\theta$  be non-negative is just

$$1 + \theta(1 - 2u)(1 - 2v) \geq 0 \quad \text{for all } (u, v) \in [0, 1]^2,$$

or equivalently that

$$\theta(1 - 2u)(1 - 2v) \geq -1 \quad \text{for all } (u, v) \in [0, 1]^2.$$

By monotonicity of  $1 - 2u$ , this holds if and only if it holds for  $(u, v) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ ; i.e.

$$\theta \geq -1, \quad -\theta \geq -1, \quad -\theta \geq -1, \quad \text{and } \theta \geq -1.$$

Thus it follows that  $f_\theta$  is a density function for  $\theta \in [-1, 1]$ , or  $|\theta| \leq 1$ .

(b) The corresponding distribution function  $F_\theta$  is given by

$$\begin{aligned} F_\theta(u, v) &= \int_0^u \int_0^v f_\theta(r, s) dr ds \\ &= \int_0^u \int_0^v \{1 + \theta(1 - 2r)(1 - 2s)\} dr ds \\ &= uv + \theta \int_0^u (1 - 2r) dr \int_0^v (1 - 2s) ds \\ &= uv + \theta u(1 - u)v(1 - v) \\ &= uv \{1 + \theta(1 - u)(1 - v)\}. \end{aligned}$$

Note that

$$F_\theta(u, 1) = u, \quad \text{and } F_\theta(1, v) = v,$$

so  $F_\theta$  has Uniform(0, 1) marginal distributions.

(c) It follows from part (iv) of Proposition 1.4.1, page 20, Chapter 1, that (by taking  $G(x) = x$ ,  $H(x) = x$ )

$$\begin{aligned} \text{Cov}(U, V) &= \int_0^1 \int_0^1 \{F_\theta(u, v) - uv\} dudv \\ &= \int_0^1 \int_0^1 \theta u(1 - u)v(1 - v) dudv \\ &= \theta \left( \int_0^1 u(1 - u) du \right)^2 \\ &= \frac{1}{36} \theta \end{aligned}$$

since

$$\int_0^1 u(1 - u) du = \frac{1}{2}u^2 - \frac{1}{3}u^3 \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Now since  $Var(U) = Var(V) = 1/12$  (since they are both  $Uniform(0, 1)$ ), it follows that

$$\rho(U, V) = \frac{Cov(U, V)}{\sqrt{Var(U)Var(V)}} = \frac{\theta/36}{\sqrt{(1/12)(1/12)}} = \frac{\theta}{3}.$$

Note that this implies that  $|\rho(U, V)| \leq 1/3$ , and hence this family of distributions does not include any distributions on  $[0, 1]^2$  with correlations larger than  $1/3$  in absolute value.

4. Suppose that  $F$  is the distribution function of random variables  $X, Y$  with  $X \sim Uniform(0, 1)$  marginally and  $Y \sim Uniform(0, 1)$  marginally. Thus  $F(x, y) = P(X \leq x, Y \leq y)$  satisfies

$$F(x, 1) = x, \quad 0 \leq x \leq 1, \quad \text{and} \quad F(1, y) = y, \quad 0 \leq y \leq 1.$$

(a) Show that

$$F(x, y) \leq x \wedge y \equiv F_U(x, y)$$

for all  $0 \leq x \leq 1, 0 \leq y \leq 1$ . Here  $x \wedge y \equiv x$  if  $x \leq y$ ,  $y$  if  $y \leq x$ .

(b) Show that

$$F(x, y) \geq (x + y - 1)^+ \equiv F_L(x, y)$$

for all  $0 \leq x \leq 1, 0 \leq y \leq 1$ . Here  $z^+ = z1_{[0, \infty)}(z)$ .

(c) Show that  $F_U$  is the distribution function of  $(X, X)$  where  $X \sim Uniform(0, 1)$ . Show that  $F_L$  is the distribution function of  $(X, 1 - X)$  where  $X \sim Uniform(0, 1)$ .

(d) The distribution functions  $F_U$  and  $F_L$  are called the Fréchet bounds. Show that  $F_L$  and  $F_U$  are singular with respect to Lebesgue measure  $\lambda_2$  on  $[0, 1]^2$ ; i.e. show that the corresponding probability measures  $P_L$  and  $P_U$  satisfy

$$P((X, Y) \in A) = 1, \quad \lambda_2(A) = 0$$

and

$$P((X, Y) \in A^c) = 0, \quad \lambda_2(A^c) = 1$$

for some set  $A$  (which will be different for  $P_L$  and  $P_U$ ). This implies that  $F_L$  and  $F_U$  do not have densities with respect to Lebesgue measure on  $[0, 1]^2$ . (See Chapter 0, Section 3, especially Definition 3.1 and Theorem 3.1.)

**Solution:** (a) By monotonicity of  $F$  in each argument,

$$F(x, y) \leq F(x, 1) = x \quad \text{and} \quad F(x, y) \leq F(1, y)$$

for  $0 \leq x, y \leq 1$ . Hence it follows that

$$F(x, y) \leq \min\{x, y\} \equiv F_U(x, y), \quad 0 \leq x, y \leq 1.$$

(b) Note that

$$0 \leq P(X > x, Y > y) = 1 - F(x, 1) - F(1, y) + F(x, y) = 1 - x - y + F(x, y),$$

and hence

$$F(x, y) \geq x + y - 1, \quad 0 \leq x, y \leq 1.$$

Since  $F(x, y) = P(X \leq x, Y \leq y) \geq 0$  trivially, it follows that

$$F(x, y) \geq (x + y - 1) \vee 0 = (x + y - 1)^+ \equiv F_L(x, y) \quad 0 \leq x, y \leq 1.$$

(c) If  $X \sim \text{Uniform}(0, 1)$ , the joint distribution function of the pair  $(X, Y) = (X, X)$  is given by

$$P(X \leq x, Y \leq y) = P(X \leq x, X \leq y) = P(X \leq x \wedge y) = x \wedge y$$

for  $0 \leq x, y \leq 1$ . Thus  $F_U$  is the distribution function of  $(X, X)$ . Similarly, the joint distribution function of the pair  $(X, Y) = (X, 1 - X)$  is given by

$$\begin{aligned} P(X \leq x, 1 - X \leq y) &= P(X \leq x, X \geq 1 - y) \\ &= \begin{cases} P(1 - y \leq X \leq x) & \text{if } x \geq 1 - y \\ 0 & \text{if } x < 1 - y \end{cases} \\ &= \begin{cases} x - (1 - y) & \text{if } x \geq 1 - y \\ 0 & \text{if } x < 1 - y \end{cases} \\ &= (x + y - 1)^+. \end{aligned}$$

Thus  $F_L$  is the distribution function of  $(X, 1 - X)$ .

(d) It is clear from part (c) that if we take  $A = \{(x, x) : 0 \leq x \leq 1\} \subset [0, 1]^2$ , then  $P_U((X, Y) \in A) = 1$  and  $\lambda_2(A) = 0$ . Similarly, if  $B = \{(x, 1 - x) : 0 \leq x \leq 1\} \subset [0, 1]^2$ , then  $P_L((X, Y) \in B) = 1$  and  $\lambda_2(B) = 0$ . Thus both  $P_U$  and  $P_L$  are singular with respect

to Lebesgue measure  $\lambda_2$  on  $[0, 1]^2$ , and the hypothesis of the Radon-Nikodym theorem 0.3.1 is violated. Hence neither  $P_U$  nor  $P_L$  have densities with respect to Lebesgue measure  $\lambda_2$ . Also note that

$$\frac{\partial^2}{\partial x \partial y} F_U(x, y) = 0 \text{ for } (x, y) \notin A,$$

while

$$\frac{\partial^2}{\partial x \partial y} F_L(x, y) = 0 \text{ for } (x, y) \notin B.$$

[Note that the arguments in (a) and (b) extend to an arbitrary distribution function  $F$  on  $R^2$  with marginal d.f.'s  $F_X$  and  $F_Y$  respectively:

$$F(x, y) \leq F_X(x) \wedge F_Y(y),$$

and

$$F(x, y) \geq (F_X(x) + F_Y(y) - 1)^+ .]$$

Here are plots of the Fréchet upper and lower bounds  $F_U$  and  $F_L$  on  $[0, 1]^2$ .

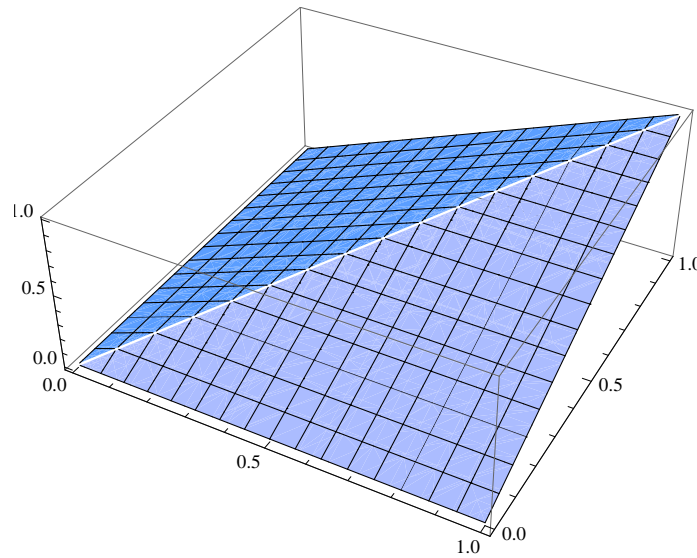


Figure 2:  $F_U$ .

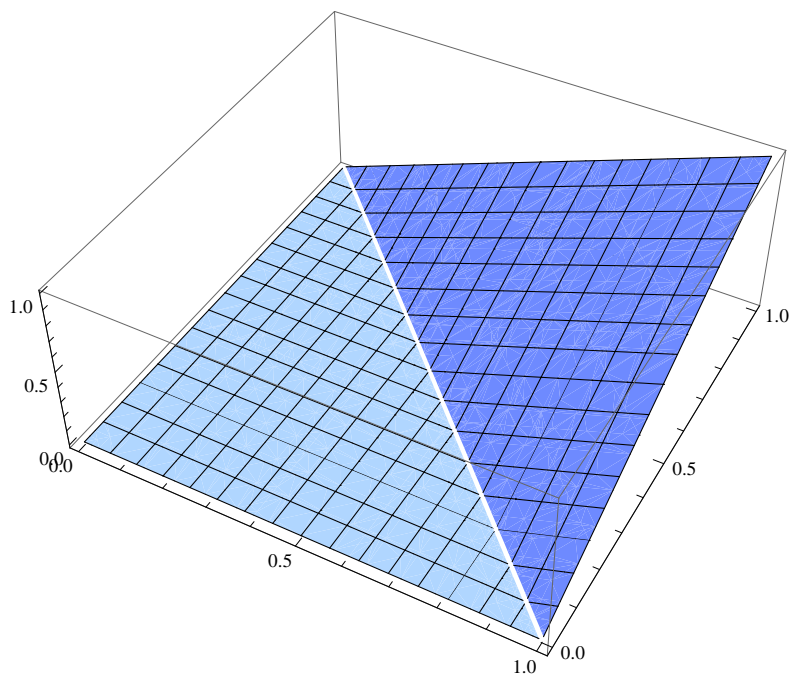


Figure 3:  $F_L$ .

5. Lehmann and Casella, TPE, problem 1.10, page 62. Show that for real numbers  $x_1, \dots, x_n$  in the domain of a monotone real-valued functions  $h$ , the function

$$H(a) \equiv \frac{1}{n} \sum_{i=1}^n (h(x_i) - h(a))^2$$

is minimized by that value of  $a$ , say  $a_{min}$ , given by

$$a_{min} = h^{-1} \left( \frac{1}{n} \sum_{i=1}^n h(x_i) \right).$$

Find particular functions  $h$  so that this yields the arithmetic, harmonic, and geometric means of the  $x_i$ 's.

**Solution:** First note that that  $H(a)$  is minimized by any value of  $a$ , say  $a_{min}$ , satisfying

$$h(a_{min}) = \frac{1}{n} \sum_{i=1}^n h(x_i).$$

This follows by noting that

$$H(a) = H(a_{min}) + (h(a_{min}) - h(a))^2 .$$

Since we have assumed that  $h$  is (strictly) monotone, it follows easily that

$$a_{min} = h^{-1} \left( \frac{1}{n} \sum_1^n h(x_i) \right) .$$

When  $h(x) = x$ ,

$$a_{min} = \frac{1}{n} \sum_1^n (x_i) = \text{the arithmetic mean} .$$

When  $h(x) = 1/x$ ,

$$a_{min} = \frac{1}{\frac{1}{n} \sum_1^n \frac{1}{x_i}} = \text{the harmonic mean} .$$

When  $h(x) = \log(x)$ ,

$$\begin{aligned} a_{min} &= \exp \left( \frac{1}{n} \sum_1^n \log(x_i) \right) = \exp \left( \frac{1}{n} \log \left( \prod_1^n x_i \right) \right) \\ &= \exp \left( \log \left( \left( \prod_1^n x_i \right)^{1/n} \right) \right) = \left( \prod_1^n x_i \right)^{1/n} \\ &= \text{the geometric mean} . \end{aligned}$$