Statistics 581

Problem Set 5

Wellner; 10/27/2010

Reading: Ferguson, ACLST, Chapters 13 and 14, pages 87 - 100;
Wellner Notes, Chapter 2, sections 4 - 6.
Due: Wednesday, November 3, 2010.
Reminder: Midterm exam, Monday, November 8, 2010

1. Suppose that $X_i \sim$ Bernoulli(p_i), $i = 1, \ldots, n$ are independent. (a) Show that if

$$\sum_{i=1}^n p_i(1 - p_i) \to \infty\,, \tag{1}$$

then

$$\frac{\sqrt{n}(\overline{X}_n - \overline{p}_n)}{\sqrt{n^{-1}\sum_{i=1}^n p_i(1 - p_i)}} \to_d N(0, 1)\,.$$

(b) Give one example $\{p_i\}_{i\ge 1}$ for which (1) holds and another example for which it fails.
(c) Compare the condition (1) to the condition for "Poisson approximation" given by $\sum_{i=1}^n p_{n,i}^2 \to 0$ given by Le Cam's inequality (see Ferguson, ACILST, probelm 5, page 18). Can both conditions hold?

2. Suppose that $X_1, \ldots, X_n$ are independent with common mean μ, but with variances $\sigma_1^2, \ldots, \sigma_n^2$ respectively.
(a) Show that $\overline{X}_n$ is a consistent estimator of μ if $\sum_{i=1}^n \sigma_i^2 = o(n^2)$.
(b) Now suppose that $X_i = \mu + \sigma_i \epsilon_i$ where $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. with some distribution function F with $E(\epsilon_1) = 0$ and $Var(\epsilon_1) = 1 < \infty$. Show that if

$$\max_{1\le i\le n} \sigma_i^2 / \sum_{i=1}^n \sigma_i^2 \to 0 \tag{2}$$

then with $\overline{\sigma}_n^2 \equiv n^{-1}\sum_1^n \sigma_i^2$,

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{\overline{\sigma}_n} \to_d N(0, 1)\,.$$

Hence show that if both (2) and

$$\overline{\sigma_n^2} \to \text{“something”} \equiv \sigma_0^2\,, \tag{3}$$

then

$$\sqrt{n}(\overline{X}_n - \mu) \to_d N(0, \sigma_0^2)\,.$$

(c) Show that (2) holds but that (3) fails if $\sigma_i^2 = Ai^r$ with $r < 1$. Hence show that in this case $n^{(1-r)/2}(\overline{X}_n - \mu) = O_p(1)$.

3. Suppose that $X_1, \ldots, X_n$ are independent with common mean μ, but with variances $\sigma_1^2, \ldots, \sigma_n^2$ respectively, exactly as in problem 2 above. Consider estimators of μ of the form $T_n \equiv T_n(w) = \sum_{i=1}^n w_{ni} X_i$ where $w = w_n = (w_{n1}, \ldots, w_{nn})$ is a vector of weights with $\sum_{i=1}^n w_{ni} = 1$.
(a) Show that all the estimators $T_n(w)$ are unbiased, and that the choice of weights which minimizes $Var(T_n(w))$ is

$$w_{ni}^{opt} = \frac{1/\sigma_i^2}{\sum_{j=1}^n (1/\sigma_j^2)} \qquad \text{for } i = 1, \ldots, n\,. \tag{4}$$

(b) Compute $Var(T_n(w^{opt}))$ and show that $T_n(w^{opt})$ is a consistent estimator of μ if $\sum_1^n (1/\sigma_j^2) \to \infty$.
(c) Now suppose that $X_i = \mu + \sigma_i \epsilon_i$ where $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. with some distribution function F with $E(\epsilon_1) = 0$ and $Var(\epsilon_1) = 1 < \infty$ as in 2(b) above. Show that

$$\sqrt{\sum_1^n (1/\sigma_i^2)} \left(T_n(w^{opt}) - \mu \right) \to_d N(0,1)$$

if

$$\frac{\max_{1 \le i \le n} (1/\sigma_i^2)}{\sum_{j=1}^n (1/\sigma_j^2)} \to 0\,.$$

(d) Compute $Var[T_n(w^{opt})]/Var[\overline{X}_n]$ in the case $\sigma_i^2 = A i^r$ for $r = .25,\ .50,\ .75,\ 1$ and $n = 5, 10, 20, 50, 100$, and ∞.

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. Cauchy$(0,1)$; so the density of each X_i with respect to Lebesgue measure on R is $f(x) = \pi^{-1}(1+x^2)^{-1}$, $x \in R$.
(a) Compute the distribution function F of the X_i's.
(b) Compute and plot the inverse distribution function F^{-1} corresponding to F.
(c) For what values of $r > 0$ is $E|X_1|^r < \infty$?
(d) Find the distribution function of $M_n \equiv \max_{1 \le i \le n} X_i$.
(e) For what values of r is $E|M_n|^r < \infty$?
(f) Find a sequence of constants b_n so that $M_n / b_n \to_d$ and find the limiting distribution. [Hint: see Ferguson, ACLST, Theorem 14, page 95.]

5. Suppose that $X_1, \ldots, X_n$ are i.i.d. random vectors with values in R^k with $E(X_1) = \mu$ and $E(X_1^T X_1) < \infty$ so that $\Sigma = E(X_1 - \mu)(X_1 - \mu)^T$ is well-defined. Thus

$$Z_n \equiv \sqrt{n}(\overline{X}_n - \mu) \to_d Z \sim N_k(0, \Sigma)\,.$$

Suppose that $g : R^k \to R$ is a function, and suppose that $\nabla g = (g')^T$ exists at μ. Then the delta-method (or g' theorem) tells us that

$$\sqrt{n}(g(\overline{X}_n) - g(\mu)) \to_d \nabla g(\mu)^T Z \sim N(0, \nabla g(\mu)^T \Sigma \nabla g(\mu))\,. \tag{5}$$

(a) Show that we can strengthen (5) as follows: Suppose that $\nabla g = (g')^T$ is continuous at μ. Then $\sqrt{n}(g(\overline{X}_n) - g(\mu))$ is *asymptotically linear* at μ:

$$\begin{aligned} \sqrt{n}(g(\overline{X}_n) - g(\mu)) &= \nabla g(\mu)^T \sqrt{n}(\overline{X}_n - \mu) + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(X_i) + o_p(1) \end{aligned}$$

where

$$\psi(x) = \nabla g(\mu)^T (x - \mu)$$

which is called the *influence function* of $g(\overline{X}_n)$ as an estimator of $g(\mu)$, has mean $E\psi(X_i) = 0$ and $Var(\psi(X_i)) = \nabla g(\mu)^T \Sigma \nabla g(\mu)$.
(b) Does the result of (a) apply to the situation considered in problem 1(b) of problem set #4? If so, what is the resulting influence function?

6. **Optional bonus problem 1.**
(a) Write out a proof of (10) on page 16 of the Chapter 2 notes.
(b) Write out a proof of the corresponding fact concerning the general empirical process $\mathbb{G}_n$: $\mathbb{G}_n \to_{f.d.} \mathbb{G}$ where $\mathbb{G}_n$ and $\mathbb{G}$ are as defined on page 21 of the chapter 2 notes; i.e. for any $f_1, \ldots, f_k \in L_2(P)$, $(\mathbb{G}_n(f_1), \ldots, \mathbb{G}_n(f_k)) \to_d (\mathbb{G}(f_1), \ldots, \mathbb{G}(f_k))$.

7. **Optional bonus problem 2.**
Suppose that $X_1, \ldots, X_n$ are i.i.d. exponential(θ); i.e. with density $p_\theta(x) = \theta \exp(-\theta x) 1_{[0,\infty)}(x)$. Let $X_{(n)} = X_{n:n}$ be the largest order statistic of $X_1, \ldots, X_n$.
(a) Find constants c_n so that $Y_n = X_{(n)} - c_n \to_d Y$ for some random variable Y and find the limiting distribution of F_Y.
(b) Compute the density of Y_n and show that it converges to the density f_Y of Y.
(c) What can you conclude from the result of (b) and Scheffé's theorem (chap. 2 notes, prop. 1.14, page 9?)