

Statistics 581, Problem Set 2

Wellner; 10/6/2010

Reading: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 1-25.

Due: Wednesday, October 13, 2010.

1. Suppose that Y is a random variable with $E(Y^2) < \infty$.
 - (a) Show that

$$\text{Var}(Y) = E\{\text{Var}(Y|X)\} + \text{Var}\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}.$$

- (b) Interpret (a) geometrically.
- (c) Suppose that $(Y|X) \sim N(-X^2/2, \theta X^2)$ where $X \sim \text{Exponential}(1)$ and $\theta > 0$. Compute $E(Y)$ and $\text{Var}(Y)$. Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

2. (a) The case $r = 1$ of Chebyshev's inequality is known as Markov's inequality and is usually written $P(|X| \geq \epsilon) \leq E(|X|)/\epsilon$ for an arbitrary random variable X and $\epsilon > 0$. For each fixed $\epsilon > 1$, find a distribution for X with $E(X) = 0$ and $E(|X|) = 1$ that gives equality in Markov's inequality.
 - (b) For an arbitrary random variable X , show that

$$P(|X| \geq \epsilon) \leq \frac{E \cosh(X) - 1}{\cosh(\epsilon) - 1}.$$

3. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
 - (ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
 - (iii) $\Sigma\mu = \mu$.Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

4. Ferguson, ACILST, #2, page 6:
 - (a) Suppose that $X_n \sim \text{Uniform}\{1/n, 2/n, \dots, n/n\}$. Show that $X_n \rightarrow_d X \sim \text{Uniform}(0, 1)$. Does $X_n \rightarrow_p X$?
 - (b) Suppose that $Y_n = \sum_{k=1}^n (k/n) 1_{[(k-1)/n, k/n)}(U)$ where $U \sim \text{Uniform}[0, 1]$. Show that $Y_n \sim \text{Uniform}\{1/n, 2/n, \dots, n/n\}$, and $Y_n \rightarrow_p U$.

5. (a) Lehmann and Casella, #3.5, page 64.
 - (b) Lehmann and Casella, #3.6, page 64.
 - (c) Lehmann and Casella, #3.7, page 64.

6. Optional Bonus Problem 1:

Let X_1, X_2, \dots be a sequence of i.i.d. random variables (with no assumptions) of any finite moments). Does X_n/n converge almost surely to zero? If so, show it; if not, give a counterexample.

7. Optional Bonus Problem 2:

Wellner, 581 Course Notes, Chapter 1, Exercise 1.2, page 4.

8. Optional Bonus Problem 3:

Wellner 581 Course Notes, Chapter 1, Exercise 4.1, page 19.