

## Statistics 581, Problem Set 1

Wellner; 9/29/2010

**Reading:** Lehmann & Casella, TPE, pages 1 - 32; Ferguson, ACILST, Chapter 1, pages 3-7; skim Chapter 0 handout; read Chapter 1 handout.

**Due:** Wednesday, October 6, 2010.

1. Let  $X$  and  $Y$  be i.i.d.  $\text{Uniform}(0,1)$  random variables. Define  $U = X - Y$ ,  $V = \max(X, Y) = X \vee Y$ .

(i) What is the range of  $(U, V)$ ?

(ii) Find the joint density function  $f_{U,V}(u, v)$  of the pair  $(U, V)$ . Are  $U$  and  $V$  independent?

2. Ferguson, ACILST, #6, page 7. (This is known as the Polya-Cantelli lemma; see Chapter 2, Proposition 2.11, page 10.)

3. Suppose that for  $\theta \in R$ ,

$$f_{\theta}(u, v) = \{1 + \theta(1 - 2u)(1 - 2v)\}1_{[0,1]^2}(u, v).$$

(a) For what values of  $\theta$  is  $f_{\theta}$  a density function on  $[0, 1]^2$ ?

(b) For the set of  $\theta$ 's you identified in (a), find the corresponding distribution function  $F_{\theta}$  and show that it has  $\text{Uniform}(0, 1)$  marginal distributions.

(c) If  $(U, V) \sim F_{\theta}$ , compute the correlation  $\rho(U, V) \equiv \rho$  as a function of  $\theta$ . Does this show any difficulty with this family of distributions as a model of dependence?

4. Suppose that  $F$  is the distribution function of random variables  $X, Y$  with  $X \sim \text{Uniform}(0, 1)$  marginally and  $Y \sim \text{Uniform}(0, 1)$  marginally. Thus  $F(x, y) = P(X \leq x, Y \leq y)$  satisfies

$$F(x, 1) = x, \quad 0 \leq x \leq 1, \quad \text{and} \quad F(1, y) = y, \quad 0 \leq y \leq 1.$$

(a) Show that

$$F(x, y) \leq x \wedge y \equiv F_U(x, y)$$

for all  $0 \leq x \leq 1, 0 \leq y \leq 1$ . Here  $x \wedge y \equiv x$  if  $x \leq y$ ,  $y$  if  $y \leq x$ .  
 (b) Show that

$$F(x, y) \geq (x + y - 1)^+ \equiv F_L(x, y)$$

for all  $0 \leq x \leq 1, 0 \leq y \leq 1$ . Here  $z^+ = z1_{[0, \infty)}(z)$ .

(c) Show that  $F_U$  is the distribution function of  $(X, X)$  where  $X \sim \text{Uniform}(0, 1)$ . Show that  $F_L$  is the distribution function of  $(X, 1 - X)$  where  $X \sim \text{Uniform}(0, 1)$ .

(d) The distribution functions  $F_U$  and  $F_L$  are called the Fréchet bounds. Show that  $F_L$  and  $F_U$  are singular with respect to Lebesgue measure  $\lambda_2$  on  $[0, 1]^2$ ; i.e. show that the corresponding probability measures  $P_L$  and  $P_U$  satisfy

$$P((X, Y) \in A) = 1, \quad \lambda_2(A) = 0$$

and

$$P((X, Y) \in A^c) = 0, \quad \lambda_2(A^c) = 1$$

for some set  $A$  (which will be different for  $P_L$  and  $P_U$ ). This implies that  $F_L$  and  $F_U$  do not have densities with respect to Lebesgue measure on  $[0, 1]^2$ . (See Chapter 0, Section 3, especially Definition 3.1 and Theorem 3.1.)

5. Lehmann and Casella, TPE, problem 1.10, page 62.
6. **Bonus problem 1:** Suppose that  $X \sim \text{Uniform}(0, 1)$  and  $Y = 1 - X$ . Find the joint distribution function  $F(x, y) = F_{X, Y}(x, y)$  of  $(X, Y)$ . Does this distribution function have a joint density with respect to Lebesgue measure?
7. **Bonus problem 2:** Lehmann and Casella, TPE, problem 3.8, page 64, modified as follows: (a) First suppose that  $X$  has distribution function  $F$  on  $\mathbb{R}^+ = [0, \infty)$  and  $Y$  with distribution function  $G$  on  $\mathbb{R}^+$  are independent random variables. Find the joint distribution of  $Z = \min\{X, Y\}$  and  $W = 1\{X \leq Y\}$ .  
 (b) Show that if  $F = \text{exponential}(\lambda)$  and  $G = \text{exponential}(\mu)$ , then  $Z$  and  $W$  are independent.