STATISTICS 581:
Day 1 Quiz Solutions, Fall, 2009

1. **State** the (Lindeberg) Central Limit Theorem.

Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$, then

$$\sqrt{n}(\overline{X}_n - \mu) \to_d Y \sim N(0, \sigma^2).$$

That is,

$$P(\sqrt{n}(\overline{X}_n - \mu) \le x) \to P(Y \le x) = P(\sigma Z \le x) = \Phi(x/\sigma)$$

where $Z \sim N(0,1)$ and $P(Z \le z) \equiv \Phi(z) = \int_{-\infty}^{z} (2\pi)^{-1/2} \exp(-t^2/2)\, dt$

2. Suppose that $U \sim$ Uniform$(0,1)$. For what values of $r \in R$ is it true that $E(U^r) < \infty$? For the values of r for which the integral is finite, compute it explicitly.

Solution:

$$E(U^r) = \int_0^1 u^r du = \frac{1}{r+1} u^{r+1} \Big|_0^1 = \frac{1}{r+1} \qquad \text{if} \;\; r > -1.$$

If $r \le -1$, then $E(U^r) = \int_0^1 u^r du = \infty$.

3. Suppose that $U \sim$ Uniform$(0,1)$ and $V_n = j/n$ on the set $[(j-1)/n \le U < j/n]$ for $j = 1, \ldots, n$. What is the distribution of V_n?

Solution: We compute

$$P(V_n = j/n) = P((j-1)/n \le U < j/n) = (j/n) - (j-1)/n = 1/n$$

for $j = 1, \ldots, n$. That is V_n has a discrete uniform distribution on the set $\{1/n, 2/n, \ldots, n/n\}$; i.e. $V_n \sim$ Uniform$\{1/n, 2/n, \ldots, n/n\}$.

4. **Define** what is meant by:
(a) V_n converges in distribution to V for random variables V and V_n, $n \ge 1$.
(b) V_n converges in probability to V.
(c) Does the sequence V_n defined in problem 3 converge in distribution? Does it converge in probability?

Solution: (a) $V_n \to_d V$ if

$$F_{V_n}(x) \equiv P(V_n \le x) \to P(V \le x) \equiv F_V(x)$$

for all $x \in C_{F_V}$, the set of continuity points of F_V.
(b) $V_n \to_p V$ if

$$P(|V_n - V| > \epsilon) \to 0 \;\; \text{as} \;\; n \to \infty$$

for every $\epsilon > 0$.
(c) Yes, $V_n \to_p U$ as $n \to \infty$, and hence also $V_n \to_d U$ since convergence in probability implies convergence in distribution. To see that $V_n \to_p U$, let $\epsilon > 0$. Then

$$P(|V_n - U| > \epsilon) = \begin{cases} n(1/n - \epsilon) = 1 - n\epsilon\,, & \text{if } 1/n > \epsilon \\ 0\,, & \text{if } 1/n \le \epsilon\,. \end{cases}$$

Hence it follows that $V_n \to_p U$.

5. Suppose that U is a random variable with a Uniform$(0,1)$ distribution. For each integer $n \geq 2$ define $X_n = (n/\log n) 1_{[0,1/n]}(U)$.
(a) Does $X_n \to_d X$? (If so, identify the distribution of the limiting variable X.)
(b) Does $X_n \to_p X$? (If so, identify the limit variable X.)
(c) Compute $E(X_n)$. Does it converge to $E(X)$ for some X?

Solution: (a) & (b) $X_n \to_p 0 \equiv X$. To see this, let $\epsilon > 0$. Then

$$P(|X_n| > \epsilon) \leq P(U \leq 1/n) = 1/n \to 0.$$

Since $X_n \to_p 0$, we also have $X_n \to_d 0$.
(c)

$$\begin{aligned} E(X_n) &= E\{(n/\log n) 1_{[0,1/n]}(U)\} = (n/\log n) E 1_{[0,1/n]}(U) \\ &= (n/\log n) P(U \leq 1/n) = (n/\log n)(1/n) = 1/\log n \to 0. \end{aligned}$$

Thus $E(X_n) \to 0 = E(X)$ where $P(X = 0) = 1$.

6. Suppose that $X_1, \ldots, X_n, \ldots$ are independent and identically distributed Bernoulli(p) random variables (i.e. $P(X_i = 1) = p = 1 - P(X_i = 0)$ for $i = 1, 2, \ldots$). Let $T_n = X_1 + \ldots + X_n$.
(a) What is the distribution of T_n?
(b) Does $\overline{X}_n = n^{-1} T_n \to_p$ something? If so, what is "something"?
(c) Does $\sqrt{n}(\overline{X}_n - p) \to_d$ something? If so, what is "something"?
(d) What is the Cramér-Rao bound for unbiased estimators of p?

Solution: (a) $T_n \sim$ Binomial(n, p).
(b) $\overline{X}_n = n^{-1} T_n \to_p p$ by the weak law of large numbers.
(c) $\sqrt{n}(\overline{X}_n - p) \to_d N(0, p(1-p))$ by the Central Limit Theorem.
(d) The density (mass function) for a sample of size one is

$$p(x; p) = p^x (1-p)^{1-x} \quad \text{for} \quad x \in \{0, 1\}.$$

Thus

$$\log p(x; p) = x \log p + (1 - x) \log(1 - p)$$

and the score function for estimating p is given by

$$\dot{l}_p(x) = \frac{x}{p} - \frac{1-x}{1-p} = \frac{x-p}{p(1-p)}.$$

Thus the information for p in a sample of size $n = 1$ is

$$I(p) = E(\dot{l}_p(X)^2) = \frac{E(X-p)^2}{[p(1-p)]^2} = \frac{1}{p(1-p)}.$$

(Alternatively,

$$I(p) = -E(\ddot{l}_{pp}(X)) = E\left\{\frac{X}{p^2} + \frac{1-X}{(1-p)^2}\right\} = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}.)$$

It follows that the Cramér-Rao lower bound for unbiased estimates of p based on n observations is:

$$\frac{1}{nI(p)} = \frac{p(1-p)}{n}.$$