

Statistics 581, Problem Set 7 Solutions

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1. Compute and plot the *score for location*, $-(f'/f)(x)$ when:
- A. $f(x) = \phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, (normal or Gaussian);
 - B. $f(x) = \exp(-x)/(1 + \exp(-x))^2$, (logistic);
 - C. $f(x) = \frac{1}{2} \exp(-|x|)$, (double exponential);
 - D. $f = t_k$, the t -distribution with k degrees of freedom;
 - E. $f(x) = \exp(-x) \exp(-\exp(-x))$, Gumbel or extreme value.

Soluton: A. For $f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, it follows that $\log f(x) = -x^2/2 + \text{constant}$ so that $(-f'/f)(x) = x$, $-1 - x(f'/f)(x) = x^2 - 1$.

B. For $f(x) = e^{-x}/(1 + e^{-x})^2$, $\log f(x) = -x - 2 \log(1 + e^{-x})$ and

$$-\frac{f'}{f}(x) = \frac{1 - e^{-x}}{1 + e^{-x}},$$

while

$$-1 - x \frac{f'}{f}(x) = x \frac{1 - e^{-x}}{1 + e^{-x}} - 1 \sim |x| - 1 \quad \text{as} \quad |x| \rightarrow \infty.$$

C. For $f(x) = 2^{-1} \exp(-|x|)$,

$$\log f(x) = -|x| + \text{constant},$$

and

$$-\frac{f'}{f}(x) = \begin{cases} -1 & x < 0 \\ \text{undefined} & x = 0 \\ +1 & x > 0 \end{cases},$$

while

$$-1 - x \frac{f'}{f}(x) = |x| - 1, \quad \text{for} \quad x \neq 0.$$

D. For the t_k distribution, $f(x) = \frac{\Gamma(\frac{1}{2}(k+1))}{\Gamma(\frac{1}{2}k)} \frac{1}{\sqrt{\pi k}} (1 + \frac{x^2}{k})^{-(k+1)/2}$,

$$\log f(x) = -\frac{k+1}{2} \log(1 + \frac{x^2}{k}),$$

and

$$-\frac{f'}{f}(x) = \frac{k+1}{k} \frac{x}{1 + \frac{x^2}{k}},$$

while

$$-1 - x \frac{f'}{f}(x) = k \frac{x^2 - 1}{x^2 + k}.$$

E. For $f(x) = \exp(-x) \exp(-\exp(-x))$,

$$\log f(x) = -x - \exp(-x),$$

and

$$-\frac{f'}{f}(x) = 1 - \exp(-x),$$

while

$$-1 - x\frac{f'}{f}(x) = -1 + x(1 - \exp(-x)).$$

Plots of these score functions for location are given in Figure 1. Note that they are *odd functions* in cases A-D, which are all symmetric densities about zero. In case E, corresponding to the asymmetric extreme value density, the score for location does not have any obvious symmetry property.

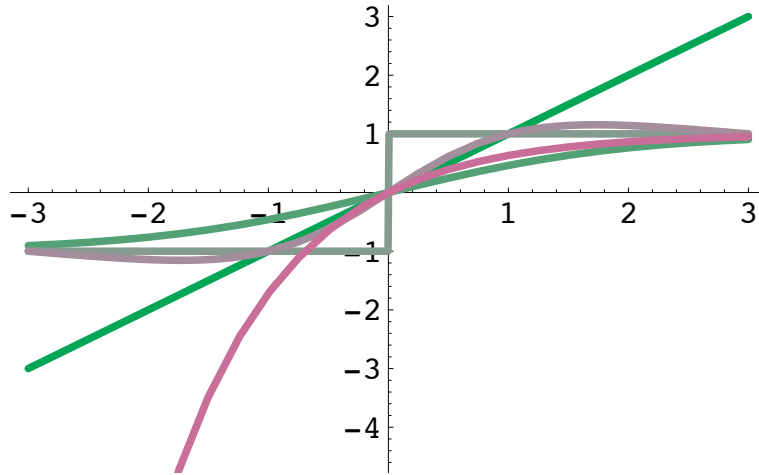


Figure 1: Scores for location.

2. Compute $I_f = \int (f'(x)/f(x))^2 f(x) dx$, the information for location, for each of the densities in problem 1.

Solution: A. In this case $I_f = \int x^2 \phi(x) dx = \text{Var}(Z) = 1$ where $Z \sim N(0, 1)$.

B. For the logistic density the information for location is

$$\begin{aligned} I_f &= \int_{-\infty}^{\infty} \left(\frac{1 - e^{-x}}{1 + e^{-x}} \right)^2 dF(x) = \int_{-\infty}^{\infty} (2F(x) - 1)^2 dF(x) \\ &= \int_0^1 (2u - 1)^2 du = 4\text{Var}(U) = 4 \frac{1}{12} = \frac{1}{3}. \end{aligned}$$

C. For the double-exponential density, $[(-f'/f)(x)]^2 = 1$, so $I_f = 1$.

D. For the t - distribution with k degrees of freedom, by using a change of variables and letting T_r denote a random variable with the t - distribution with r degrees of freedom,

$$I_f = \int_{-\infty}^{\infty} \left(\frac{k+1}{k} \right)^2 \frac{x^2}{(1+x^2/k)^2} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{k}{2})\sqrt{\pi k}} \frac{1}{(1+x^2/k)^{(k+1)/2}} dx$$

$$\begin{aligned}
&= \frac{(k+1)(k+2)}{(k+4)(k+3)} \text{Var}(T_{k+4}) \\
&= \frac{(k+1)(k+2)k+4}{(k+4)(k+3)k+2} = \frac{k+1}{k+3}
\end{aligned}$$

since $\text{Var}(T_r) = r/(r-2)$ for $r > 2$.

E. For the extreme value distribution $F(x) = \exp(-\exp(-x))$ and therefore if $X \sim F$, the random variable $Y \equiv \exp(-X) \sim \text{exponential}(1)$:

$$\begin{aligned}
P(Y \geq y) &= P(\exp(-X) \geq y) = P(X \leq -\log(y)) \\
&= \exp(-\exp(\log(y))) = \exp(-y).
\end{aligned}$$

Since $-(f'/f)(x) = -1 + e^{-x}$, it is easy to see that

$$I_f = E\left[-\frac{f'}{f}(X)\right]^2 = E[\exp(-X) - 1]^2 = E[Y - 1]^2 = \text{Var}(Y) = 1.$$

3. Consider the two parameter location-scale model

$$\mathcal{P} = \{P_\theta : \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta\}$$

where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right),$$

and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ .

(a) Calculate the information matrix $I(\theta)$ for θ .

(b) For which of the densities in A-E of problem 1 is $I_{12}(\theta)$ not zero?

Solution: (a) The score for location is

$$\begin{aligned}
\dot{\mathbf{i}}_1(x) &= \frac{\partial}{\partial \theta_1} \log \left\{ \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) \right\} \\
&= -\frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \frac{1}{\theta_2}
\end{aligned}$$

and the score for scale is

$$\begin{aligned}
\dot{\mathbf{i}}_2(x) &= \frac{\partial}{\partial \theta_2} \log \left\{ \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) \right\} \\
&= -\frac{1}{\theta_2} - \frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \frac{(x - \theta_1)}{\theta_2^2}
\end{aligned}$$

Thus we compute

$$I_{11}(\theta) = E\dot{\mathbf{i}}_1^2(X) = \frac{1}{\theta_2^2} \int \left(\frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right)\right)^2 \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) dx$$

$$\begin{aligned}
&= \frac{1}{\theta_2^2} \int \left(\frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{f,loc}, \\
I_{22}(\theta) &= E\dot{\mathbf{i}}_2^2(X) = \frac{1}{\theta_2^2} \int \left(-1 - \frac{(x - \theta_1)}{\theta_2} \frac{f'}{f} \left(\frac{x - \theta_1}{\theta_2} \right) \right)^2 \frac{1}{\theta_2} f \left(\frac{x - \theta_1}{\theta_2} \right) dx \\
&= \frac{1}{\theta_2^2} \int \left(-1 - y \frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{f,scale}, \\
I_{12}(\theta) &= E\dot{\mathbf{i}}_1(X)\dot{\mathbf{i}}_2(X) = I_{21}(\theta) \\
&= \frac{1}{\theta_2^2} \int \left(-\frac{f'}{f} \left(\frac{x - \theta_1}{\theta_2} \right) \right) \left(-1 - \frac{(x - \theta_1)}{\theta_2} \frac{f'}{f} \left(\frac{x - \theta_1}{\theta_2} \right) \right) \frac{1}{\theta_2} f \left(\frac{x - \theta_1}{\theta_2} \right) dx \\
&= \frac{1}{\theta_2^2} \int y \left(\frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{12,f}.
\end{aligned}$$

(b) Note that $I_{12,f} = 0$ for all symmetric densities f since y is odd while $(f'/f)^2 f$ is even. Thus $I_{12}(\theta) = 0$ for cases A-D in problem 1, while $I_{12}(\theta) = \theta_2^{-2} I_{12,f} \neq 0$ in case E. I calculate

$$\begin{aligned}
I_{12,f} &= \int y \left(\frac{f'}{f}(y) \right)^2 f(y) dy \\
&= \int_{-\infty}^{\infty} y (-1 + e^{-y})^2 e^{-y} \exp(-e^{-y}) dy \\
&= - \int_0^{\infty} (\log v) (v - 1)^2 e^{-v} dv = -(1 - \gamma)
\end{aligned}$$

where γ is Euler's constant. In fact this is related to I_{12} that we calculated already for the Weibull family in Example 3.2.5 in the notes and in the first display at the top of page 2 of the Handout on Gamma, Digamma, and Polygamma.

4. Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty) \equiv \Theta.$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0 ; here $F_\theta(x) \equiv P_\theta(X \leq x)$.

- (i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page 127, Table 6.1
- (ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
- (iii) Find the efficient influence functions for estimation of q_A and q_B .
- (iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators \bar{X}_n and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.

Solution: For the $\text{Gamma}(\alpha, \beta)$ parametrized my way:

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x).$$

Thus

$$\log p_\theta(x) = (\alpha - 1) \log x + \alpha \log \beta - \log \Gamma(\alpha) - \beta x,$$

and hence

$$\begin{aligned} \dot{l}_\alpha(x) &= \log x + \log \beta - \frac{\Gamma'}{\Gamma}(\alpha) = \log(\beta x) - \psi(\alpha), \\ \dot{l}_\beta(x) &= \frac{\alpha}{\beta} - x \end{aligned}$$

Furthermore,

$$\begin{aligned} \ddot{l}_{\alpha\alpha}(x) &= -\psi'(\alpha), \\ \ddot{l}_{\alpha\beta}(x) &= \frac{1}{\beta} = \ddot{l}_{\beta\alpha}(x), \\ \ddot{l}_{\beta\beta}(x) &= -\frac{\alpha}{\beta^2}. \end{aligned}$$

Hence

$$I(\theta) = \begin{pmatrix} \psi'(\alpha) & -1/\beta \\ -1/\beta & \alpha/\beta^2 \end{pmatrix}.$$

(ii). Now $q_A(\theta) = \alpha/\beta$, and

$$\begin{aligned} q_B(\theta) = P_\theta(X \leq x_0) &= \int_0^{x_0} \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} dx \\ &= \int_0^{\beta x_0} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t} dt \\ &\equiv \frac{\Gamma(\alpha, \beta x_0)}{\Gamma(\alpha)} \end{aligned}$$

where $\Gamma(\alpha, y)$ is the incomplete gamma function; note that $\Gamma(\alpha, \infty) = \Gamma(\alpha)$. Therefore

$$\begin{aligned} \dot{q}_A^T(\theta) &= \left(\frac{\partial}{\partial \alpha} q_A, \frac{\partial}{\partial \beta} q_B \right) = \left(\frac{1}{\beta}, -\frac{\alpha}{\beta^2} \right) = \frac{1}{\beta} \left(1, -\frac{\alpha}{\beta} \right) \\ &= \text{Cov}_\theta(X - E_\theta(X), \dot{l}_\theta^T(X)), \end{aligned}$$

while, with

$$\psi(\alpha, y) \equiv \frac{\partial}{\partial \alpha} \log \Gamma(\alpha, y) \equiv \Gamma'(\alpha, y)/\Gamma(\alpha, y),$$

$$\begin{aligned} \dot{q}_B^T(\theta) &= \left(\frac{\Gamma'(\alpha, \beta x_0)}{\Gamma(\alpha)} - \frac{\Gamma(\alpha, \beta x_0)\Gamma'(\alpha)}{\Gamma^2(\alpha)}, \frac{(\beta x_0)^{\alpha-1} e^{-\beta x_0} x_0}{\Gamma(\alpha)} \beta \right) \\ &= \left(\frac{\Gamma(\alpha, \beta x_0)}{\Gamma(\alpha)} \{ \psi(\alpha, \beta x_0) - \psi(\alpha) \}, \frac{x_0}{\beta} p_\theta(x_0) \right) \\ &= (q_B(\theta) \{ \psi(\alpha, \beta x_0) - \psi(\alpha) \}, \frac{x_0}{\beta} p_\theta(x_0)) \\ &= \text{Cov}_\theta[(1_{[0, x_0]}(X) - F_\theta(x_0)), \dot{l}_\theta^T]. \end{aligned}$$

(iii). The scores are given by

$$\dot{l}_\theta(x) = \begin{pmatrix} \dot{l}_\alpha(x) \\ \dot{l}_\beta(x) \end{pmatrix} = \begin{pmatrix} \log(\beta x) - \psi(\alpha) \\ \frac{\alpha}{\beta} - x \end{pmatrix}$$

and the information matrix is

$$I(\theta) = \begin{pmatrix} \psi'(\alpha) & -1/\beta \\ -1/\beta & \alpha/\beta^2 \end{pmatrix}.$$

Thus

$$I^{-1}(\theta) = \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1},$$

and the efficient influence function for estimation of q_A is

$$\begin{aligned} \tilde{l}_A &= \dot{q}_A(\theta)^T I^{-1}(\theta) \dot{l}_\theta \\ &= \frac{1}{\beta} \left(1, -\frac{\alpha}{\beta}\right) \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1} \begin{pmatrix} \log(\beta x) - \psi(\alpha) \\ \frac{\alpha}{\beta} - x \end{pmatrix} \\ &= \frac{\beta}{\alpha\psi'(\alpha) - 1} \left\{ 0 \cdot (\log(\beta x) - \psi(\alpha)) + \left(\frac{1}{\beta} - \frac{\alpha}{\beta}\psi'(\alpha)\right) \left(\frac{\alpha}{\beta} - x\right) \right\} \\ &= \left(x - \frac{\alpha}{\beta}\right). \end{aligned}$$

Note that $X - E_\theta(X) \in [\dot{l}_\theta] = \dot{\mathcal{P}}$; in fact, $X - E_\theta(X) = -\dot{l}_\beta(X)$.

Similarly, $\tilde{l}_B(x) = \dot{q}_B(\theta) I^{-1}(\theta) \dot{l}_\theta(x)$; unfortunately, this does not simplify much, largely due to the fact that $1_{[0, x_0]}(X) - F_\theta(x_0) \notin [\dot{l}_\theta] = \dot{\mathcal{P}}$.

(iii) The information bound for estimation of q_A is

$$\begin{aligned} I^{-1}(P|q_A, \mathcal{P}) &= \dot{q}_A^T I^{-1}(\theta) \dot{q}_A \\ &= \frac{1}{\beta} \left(1, -\frac{\alpha}{\beta}\right) \begin{pmatrix} \alpha/\beta^2 & 1/\beta \\ 1/\beta & \psi'(\alpha) \end{pmatrix} \frac{\beta^2}{\alpha\psi'(\alpha) - 1} \begin{pmatrix} 1 \\ -\alpha/\beta \end{pmatrix} \frac{1}{\beta} \\ &= \frac{\alpha}{\beta^2} = \text{Var}_\theta(X). \end{aligned}$$

Similarly,

$$I^{-1}(P|q_B, \mathcal{P}) = \dot{q}_B^T I^{-1}(\theta) \dot{q}_B,$$

which does not simplify appreciably because $1_{[0, x_0]}(X) - F_\theta(x_0) \notin [\dot{l}_\theta] = \dot{\mathcal{P}}$. However, since we know that $\tilde{l}_B = \Pi(1_{[0, x_0]}(x) - F_\theta(x_0) | \mathcal{P})$, it follows easily that

$$I^{-1}(P|q_B, \mathcal{P}) < E_\theta(1_{[0, x_0]}(X) - F_\theta(x_0))^2 = F_\theta(x_0)(1 - F_\theta(x_0));$$

i.e. it is possible to improve on the natural nonparametric estimator $\mathbb{F}_n(x_0)$ of $q_B(\theta) = F_\theta(x_0)$ when the model holds.

5. Lehmann and Casella, TPE, Problem 6.6, page 142: if $p(x) = (1 - \epsilon)\phi(x - \xi) + (\epsilon/\tau)\phi((x - \xi)/\tau)$ where ϕ is the standard normal density, find $I(\epsilon, \xi, \tau)$.

Solution: I would prefer to write $\theta = (\xi, \epsilon, \tau)$. The first thing to note about the model

$$\mathcal{P} = \{P_\theta : (dP_\theta/d\lambda)(x) = p(x; \theta) = p(x)\}$$

is that it is a location family: $p(x; \xi, \epsilon, \tau) = p_0(x - \xi; \epsilon, \tau)$ where

$$p_0(x; \epsilon, \tau) = (1 - \epsilon)\phi(x) + (\epsilon/\tau)\phi(x/\tau).$$

Moreover, all the distributions in the family $\mathcal{P}_0 = \{P_{0,\epsilon,\tau} : \epsilon \in [0,1], \tau > 0\}$ are *symmetric about 0*. Here are plots of these densities for $\tau = 3$ and $\epsilon = 0, .2, .4, .6, .8, 1$; the second plot is of the densities p_0 for $\epsilon = .2$ and $\tau = 1.5, 4, 8, 16, 32, 64$.

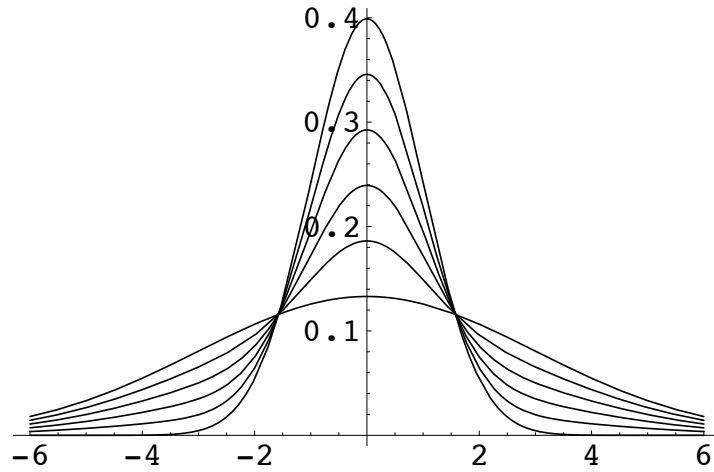


Figure 2: Densities $p(x)$ for $\xi = 0$, $\tau = 3$, $\epsilon = 0, .2, .4, .6, .8, 1.0$

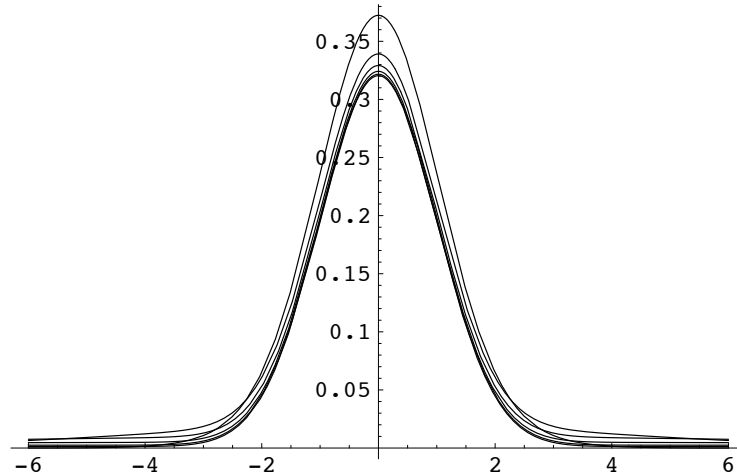


Figure 3: Densities $p(x)$ for $\xi = 0$, $\epsilon = .2$, $\tau = 1.5, 4, 8, 16, 32, 64$

Since $\phi'(x) = -x\phi(x)$, the score functions for ϵ , ξ , and τ are given by

$$l_{\xi}(x) = \frac{1}{p(x)} \left\{ (x - \xi)(1 - \epsilon)\phi(x - \xi) + \frac{x - \xi}{\tau^2} \frac{\epsilon}{\tau} \phi\left(\frac{x - \xi}{\tau}\right) \right\},$$

$$\begin{aligned} \dot{l}_\epsilon(x) &= \frac{1}{p(x)} \left\{ \frac{1}{\tau} \phi\left(\frac{x-\xi}{\tau}\right) - \phi(x-\xi) \right\}, \\ \dot{l}_\tau(x) &= \frac{1}{p(x)} \frac{\epsilon}{\tau^2} \phi\left(\frac{x-\xi}{\tau}\right) \left\{ \left(\frac{x-\xi}{\tau}\right)^2 - 1 \right\}. \end{aligned}$$

Thus with $\theta \equiv (\xi, \epsilon, \tau)$ and \dot{l}_θ , the information matrix $I(\xi, \epsilon, \tau) = I(\theta)$ is given by

$$\begin{aligned} I(\theta) &= E_\theta \{ \dot{l}_\theta(X) \dot{l}_\theta(X)^T \} \\ &= \left(E_\theta (\dot{l}_i(X) \dot{l}_j(X)) \right). \end{aligned}$$

Hence it becomes clear that all of the elements of $I(\epsilon, \xi, \tau)$ are constant functions of ξ ; hence it suffices to compute the information matrix for $\xi = 0$. Thus we take $\xi = 0$ in the rest of the argument. Now note that the scores for ϵ and τ are even functions of x : $\dot{l}_\epsilon(-x) = \dot{l}_\epsilon(x)$ and similarly for \dot{l}_τ . On the other hand, the score function for ξ is an odd function of x : $\dot{l}_\xi(-x) = -\dot{l}_\xi(x)$. It follows that

$$E_\theta \{ \dot{l}_\xi(X) \dot{l}_\epsilon(X) \} = 0, \quad \text{and} \quad E_\theta \{ \dot{l}_\xi(X) \dot{l}_\tau(X) \} = 0.$$

Thus the only non-zero entry off the diagonal is $I_{23}(\theta) = I_{\epsilon, \tau}(\theta)$. I do not know any ‘‘closed form’’ results for $I_{11}(\theta)$, $I_{22}(\theta)$, $I_{33}(\theta)$, or $I_{23}(\theta)$, but it is not hard to compute them as functions of $\theta = (\epsilon, \xi, \tau)$ especially since they depend only on (ϵ, τ) . See Figures 3-5 below.

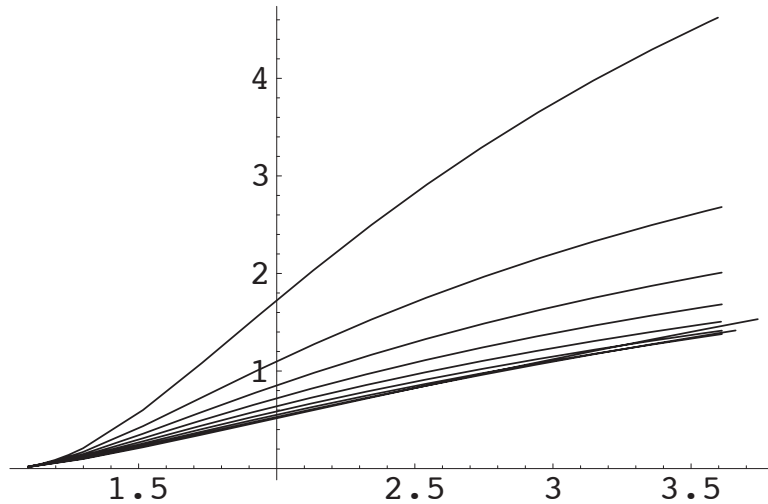


Figure 4: Information for ϵ as a function of τ for $\epsilon = .1, .2, .3, \dots, .9$

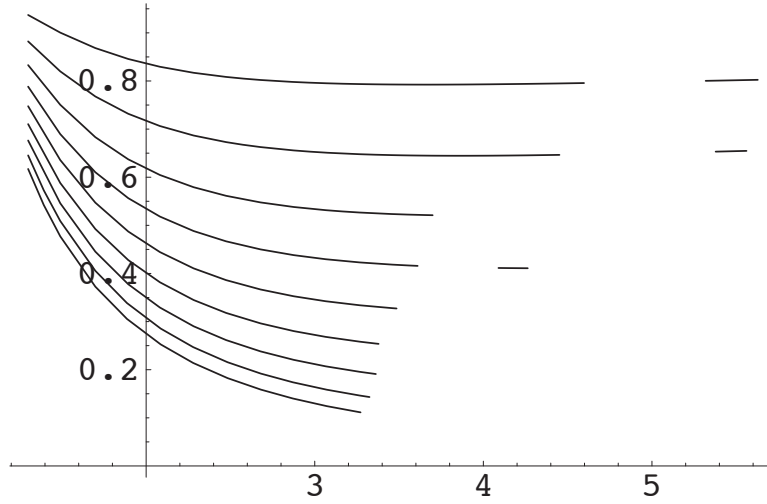


Figure 5: Information for ξ as a function of τ for $\epsilon = .1, .2, .3, \dots, .9$

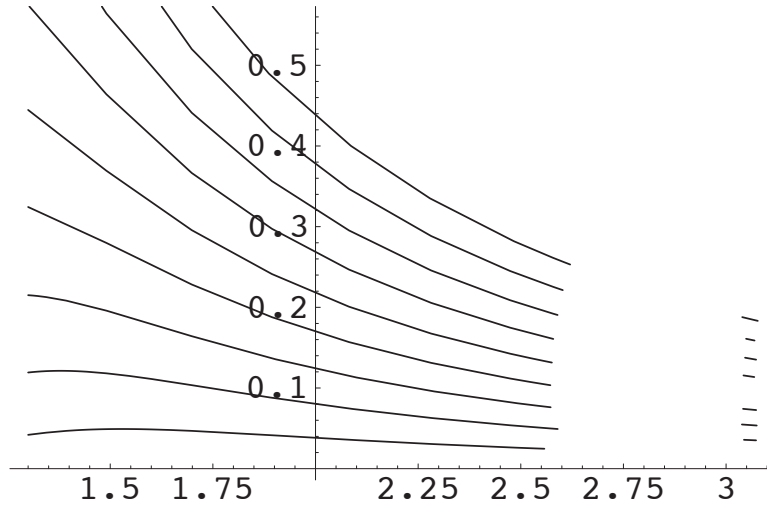


Figure 6: Information for τ as a function of τ for $\epsilon = .1, .2, .3, \dots, .9$

6. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \leq m < k$. Let $\dot{l} = \dot{l}_\theta = (\dot{l}_1, \dot{l}_2)$ be the corresponding partition of the (vector of) scores \dot{l} , and, with $\tilde{l} \equiv I^{-1}(\theta)\dot{l}$, the *efficient influence function* for θ , let $\tilde{l} = (\tilde{l}_1, \tilde{l}_2)$ be the corresponding partition of \tilde{l} . In both cases, \dot{l}_1, \tilde{l}_1 are m -vectors of functions, and \dot{l}_2, \tilde{l}_2 are $k - m$ vectors. Partition $I(\theta)$

and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where I_{11} is $m \times m$, I_{12} is $m \times (k - m)$, I_{21} is $(k - m) \times m$, I_{22} is $(k - m) \times (k - m)$. Also write

$$I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}.$$

Verify that:

A. $I^{11} = I_{11.2}^{-1}$ where $I_{11.2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21}$,

$I^{22} = I_{22.1}^{-1}$ where $I_{22.1} \equiv I_{22} - I_{21}I_{11}^{-1}I_{12}$,

$I^{12} = -I_{11.2}^{-1}I_{12}I_{22}^{-1}$,

$I^{21} = -I_{22.1}^{-1}I_{21}I_{11}^{-1}$.

This amounts to formulas (3) and (4) of section 3.2, page 14.

B. Verify that

$\tilde{l}_1 = I^{11}l_1 + I^{12}l_2 = I_{11.2}^{-1}(l_1 - I_{12}I_{22}^{-1}l_2)$, and

$\tilde{l}_2 = I^{21}l_1 + I^{22}l_2 = I_{22.1}^{-1}(l_2 - I_{21}I_{11}^{-1}l_1)$.

Solution: A. This is just block inversion/multiplication of matrices:

$$\begin{aligned} \begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} &= \begin{pmatrix} I_{11.2}^{-1} & -I_{11.2}^{-1}I_{12}I_{22}^{-1} \\ -I_{22.1}^{-1}I_{21}I_{11}^{-1} & I_{22.1}^{-1} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_{11.2}^{-1}(I_{11} - I_{12}I_{22}^{-1}I_{21}) & I_{11.2}^{-1}(I_{12} - I_{12}I_{22}^{-1}I_{22}) \\ I_{22.1}^{-1}(-I_{21} + I_{21}) & I_{22.1}^{-1}(-I_{21}I_{11}^{-1}I_{12} + I_{22}) \end{pmatrix} \\ &= \begin{pmatrix} Ident & 0 \\ 0 & Ident \end{pmatrix} = Identity. \end{aligned}$$

by using the definition of $I_{11.2}$ and $I_{22.1}$.

B. This follows immediately from the formulas for I^{11} and I^{12} by just plugging into the formula $\tilde{\mathbf{l}}_1 = I^{11}\mathbf{l}_1 + I^{12}\mathbf{l}_2$ for $\tilde{\mathbf{l}}_1$:

$$\begin{aligned} \tilde{\mathbf{l}}_1 &= I_{11.2}^{-1}\mathbf{l}_1 - I_{11.2}^{-1}I_{12}I_{22}^{-1}\mathbf{l}_2 \\ &= I_{11.2}^{-1}(\mathbf{l}_1 - I_{12}I_{22}^{-1}\mathbf{l}_2) = I_{11.2}^{-1}\mathbf{l}_1^*. \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{\mathbf{l}}_2 &= -I_{22.1}^{-1}I_{21}I_{11}^{-1}\mathbf{l}_1 + I_{22.1}^{-1}\mathbf{l}_2 \\ &= I_{22.1}^{-1}(\mathbf{l}_2 - I_{21}I_{11}^{-1}\mathbf{l}_1) = I_{22.1}^{-1}\mathbf{l}_2^*. \end{aligned}$$