**Statistics 581, Problem Set 2, Revised Solutions**
Wellner; 10/14/2009

1. Suppose that $Y$ is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}\,;$$

i.e.

$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}\,.$$

(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi^2_n(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

**Solution:** (a) We compute directly:

$$\begin{aligned} Var(Y) &= E[Y - E(Y)]^2 = E[Y - E(Y|X) + E(Y|X) - E(Y)]^2 \\ &= E[Y - E(Y|X)]^2 + 2E[(Y - E(Y|X)][E(Y|X) - E(Y)] \\ &\qquad + E[E(Y|X) - E(Y)]^2 \\ &= E\{E\{[Y - E(Y|X)]^2|X\}\} + 0 + Var[E(Y|X)] \\ &= E\{Var[Y|X]\} + Var[E(Y|X)] \end{aligned}$$

since, by computing conditionally,

$$\begin{aligned} &E[(Y - E(Y|X)][E(Y|X) - E(Y)] \\ &\quad = E\{E\{[(Y - E(Y|X)][E(Y|X) - E(Y)]|X\}\} \\ &\quad = E\{[E(Y|X) - E(Y)]E\{[Y - E(Y|X)]|X\}\} \\ &\quad = E\{[E(Y|X) - E(Y)]\{E(Y|X) - E(Y|X)\}\} \\ &\quad = E\{[E(Y|X) - E(Y)]\cdot 0\} \\ &\quad = 0\,. \end{aligned}$$

(b) A geometric interpretation of (a) is that $Y - E(Y|X)$ is orthogonal to $E(Y|X) - E(Y)$ in $L_2(\Omega, \mathcal{A}, P) = L_2(P)$, thus the identity in (a) can be intepreted as a "pythagorean theorem". Also note that $Y - E(Y|X)$ is orthogonal to any function $g(X)$: much as in the last part of (a)

$$\begin{aligned} &E\{[(Y - E(Y|X)]g(X)\} \\ &\quad = E\{E\{[(Y - E(Y|X)]g(X)|X\}\} \\ &\quad = E\{g(X)E\{[Y - E(Y|X)]|X\}\} \\ &\quad = E\{g(X)\{E(Y|X) - E(Y|X)\}\} \\ &\quad = E\{g(X)\cdot 0\} \\ &\quad = 0\,. \end{aligned}$$

(c) Now $(Y|K) \sim \chi^2_{2K+n}$ where $K \sim \text{Poisson}(\delta/2)$, so

$$E(Y) = E\{E(Y|K)\} = E\{2K+n\} = n + 2(\delta/2) = n + \delta\,.$$

Furthermore, using part (a) we get

$$\begin{aligned} Var(Y) &= E\{Var(Y|K)\} + Var\{E(Y|K)\} \\ &= E\{2(2K+n)\} + Var\{2K+n\} \\ &= 4(\delta/2) + 2n + 4(\delta/2) \\ &= 2n + 4\delta\,. \end{aligned}$$

2. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where $\Sigma$ is of rank $k < n$;
(ii) $\Sigma$ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi^2_k(\delta)$ with $\delta = \mu'\mu$.

**Solution:** See Ferguson, ACILST, page 63 (and page 57). Find $\Gamma$ orthogonal so that $\Gamma'\Sigma\Gamma = D$ where $D$ is diagonal. Now $\Gamma\Gamma' = I$, so if $\Sigma^2 = \Sigma$ we have $D^2 = \Gamma'\Sigma\Gamma\Gamma'\Sigma\Gamma = \Gamma'\Sigma^2\Gamma = \Gamma'\Sigma\Gamma = D$ and conversely. Moreover, since $\Sigma$ is of rank $k$, $D$ is of rank $k$, and this together with $D^2 = D$ implies that $D$ has $k$ 1's on the diagonal and $n-k$ 0's. Without loss, assume that $\Gamma$ has been chosen so that the $k$ ones occur in the the first $r$ positions of the diagonal matrix $D$; thus

$$D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where $I$ is $k \times k$. Moreover, note that

$$\begin{aligned} D\Gamma'\mu &= \Gamma'\Sigma\Gamma\Gamma'\mu = \Gamma'\Sigma\mu \\ &= \Gamma'\mu, \end{aligned}$$

and this implies that the last $n-k$ components of $\Gamma'\mu$ are all zero. Now let $Y = \Gamma'X$ (much as in the proof of theorem 1.3.2 of the notes). Then $Y \sim N_n(\Gamma'\mu, D)$, $Y'Y = X'\Gamma\Gamma'X = X'X$, and by (d) of page 16, section 1.3, $X'X = Y'Y \sim \chi^2_k(\delta)$ where $\delta = (\Gamma'\mu)'(\Gamma'\mu) = \mu'\Gamma\Gamma'\mu$.

3. Ferguson, ACILST, #1, page 11.
Let $X_1, X_2, \ldots$ be i.i.d. random variables with densities $f(x) = \alpha x^{-(\alpha+1)} 1_{(1,\infty)}(x)$.
(a) For what values of $\alpha > 0$ and $r > 0$ is it true that $n^{-1} X_n \to_r 0$?
(b) For what values of $\alpha > 0$ is it true that $n^{-1} X_n \to_{a.s.} 0$?
(c) If $X_1, X_2, \ldots$ are independent with $X_n$ having density $f_n(x) = \alpha_n x^{-(\alpha_n+1)} 1_{(1,\infty)}(x)$ for $n = 1, 2, \ldots$, Find the limit of $n^{-2} E X_n^2$ when $\alpha_n = 2 + n^{-\gamma}$ for $\gamma \in \mathbb{R}$.

**Solution:** (a) $E(X_n^r) = \alpha \int_1^\infty x^r x^{-(\alpha+1)} dx = \alpha/(\alpha - r)$ if $\alpha > r$, while $E(X_n^r) = \infty$ if $\alpha \leq r$. Thus $E[(n^{-1} X_n)^r] = n^{-r}(\alpha/(\alpha - r)) \to 0$ if $\alpha > r$.
(b) $P(n^{-1} X_n > \epsilon) = P(X_n > n\epsilon) = \alpha \int_{n\epsilon}^\infty x^{-(\alpha+1)} dx = (n\epsilon)^{-\alpha}$, so for $\alpha > 1$ we have

$$\sum_{n=1}^\infty P(n^{-1} X_n > \epsilon) \leq \sum_{n=1}^\infty (n\epsilon)^{-\alpha} < \infty.$$

Thus $P(n^{-1} X_n > \epsilon \text{ i.o.}) = 0$ by the Borel-Cantelli lemma, and we conclude that $n^{-1} X_n \to_{a.s.} 0$ for $\alpha > 1$. [Since the $X_n$'s are independent, $n^{-1} X_n \to_{a.s.} 0$ if *and only if* $\alpha > 1$ by the converse Borel-Cantelli lemma.]
(c) If the $X_n$'s are independent with the same densities as in (a) and (b) but with $\alpha = \alpha_n = 2 + n^{-\gamma}$ for $X_n$, then $E X_n^2 = (2 + n^{-\gamma})/n^{-\gamma}$, so

$$n^{-2} E X_n^2 = (2 + n^{-\gamma})/n^{2-\gamma} \to \begin{cases} 0, & \text{if } \gamma < 2, \\ 2, & \text{if } \gamma = 2, \\ \infty, & \text{if } \gamma > 2. \end{cases}$$

4. (a) Ferguson, ACILST, #4, page 6: Give an example of random variables $X_n$ such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.
(b) Give an example of random variables $X_n$ such that $E|X_n| \to 0$ and $E|X_n|^2 \to \infty$.
(c) Give an example of a sequence of random variables $X_n$ for which $X_n \to_p 0$ but $X_n \to_{a.s.} 0$ fails.

**Solution:** (a) If $X_n = a_n$ with probability $p_n$ and $X_n = 0$ with probability $1 - p_n$, then $E(X_n) = a_n p_n$ and $E(X_n^2) = a_n^2 p_n = 1$ if $p_n = 1/a_n^2$. Then $E(X_n) = a_n/a_n^2 = 1/a_n \to 0$ if $a_n \to \infty$. Ferguson's solution on

page 173 takes $a_n = n$; the same holds for any sequence $a_n \to \infty$.
(b) Let $U \sim$ Uniform$(0,1)$, and set $X_n = n^\alpha 1_{[0,1/n]}(U)$. Then $EX_n = n^\alpha n^{-1} \to 0$ if $\alpha < 1$, while $EX_n^2 = n^{2\alpha} n^{-1} \to \infty$ if $\alpha > 1/2$. Thus the required convergences hold for all $1/2 < \alpha < 1$.
(c) Let $U \sim$ Uniform$(0,1)$. The "dancing functions" are defined by $X_{n,k} = 1_{[(k-1)/2^n, k/2^n)}(U)$, $k = 1, \ldots, 2^n$, $n = 1, 2, \ldots$. Let $\{Y_m\}_{m \geq 1}$ be defined by $Y_m = X_{n,k}$ if $m = (\sum_{j=1}^n 2^j) + k = 2^{n+1} - 2 + k$ with $1 \leq k \leq 2^n$. Then for $\epsilon \in (0,1)$,

$$P(|Y_m| > \epsilon) = P(|X_{n,k}| > \epsilon) = 2^{-n} \to 0$$

so $Y_m \to_p 0$, but for every $U(\omega) \in (0,1)$ we have $Y_m(\omega) = 1$ for infinitely many $m$'s and also $Y_m(\omega) = 0$ for infinitely many $m$'s. Hence

$$0 = \liminf Y_m < \limsup Y_m = 1 \qquad a.s.$$

and it follows that $Y_m$ does not converge to 0 almost surely.

5. (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function $f_W$, distribution function $F_W$, and inverse distribution function $F_W^{-1}$ explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that $R$ and $\Theta$ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim$ Uniform$(-\pi/2, \pi/2)$. [Note that $g(\theta) = \tan(\theta)$ is a periodic function of $\theta$ with period $\pi$: $g(\theta) = g(\theta + \pi)$ for all $\theta$. Thus the inverse function $g^{-1}(\theta) = \arctan(\theta)$ is usually taken to be the inverse of $g$ restricted to $\theta \in (-\pi/2, \pi/2)$ where $\tan(\theta)$ is strictly increasing, negative for $-\pi/2 < \theta < 0$ and positive for $0 < \theta < \pi/2$.]
(c) Use the results of (a) and (b) to show (using Theorem 2.3.1, Chapter 2 notes, page 13) how to use two independent Uniform$(0,1)$ random variables $U$ and $V$ to generate two standard normal random variables.

**Solution:** (a) If $W \sim \chi_2^2 = \text{Gamma}(1, 1/2)$, the density function is given by $f_W(w) = (1/2)e^{-w/2} 1_{[0,\infty)}$; i.e. $W \sim$ Exponential$(1/2)$. Hence the distribution function is $F_W(w) = 1 - \exp(-w/2)$ for $w \geq 0$, and the inverse distribution function is $F_W^{-1}(u) = -2\log(1-u)$.
(b) The joint density of $(X, Y)$ is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \exp(-(x^2 + y^2)/2) \qquad \text{for} (x,y) \in R^2 \,.$$

Moreover, $x = r\cos(\theta)$ and $y = r\sin(\theta)$ uniquely for $r \in (0, \infty)$ and $\theta \in (-\pi, \pi]$. However, the inverse function $(x, y) \mapsto (\sqrt{x^2 + y^2}, \arctan(y/x)) \equiv (r, \theta)$ maps both the half planes $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $\{(x, y) \in \mathbb{R}^2 : x < 0\}$ into $\{(r, \theta) : r > 0, -\pi/2 < \theta < \pi/2)\}$. Hence both of these half planes contribute separately to the joint density of $(r, \theta)$. Since the Jacobian of the transformation $(r, \theta) \mapsto (x(r, \theta), y(r, \theta))$ is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \left| \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \right| = r\cos^2(\theta) + r\sin^2(\theta) = r\,,$$

We find that the joint density of $(R, \Theta)$ is given by

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= f_{X,Y}(r\cos(\theta), r\sin(\theta))1_{(0,\infty)}(r\cos(\theta)) \\ &\quad + f_{X,Y}(r\cos(\theta + \pi), r\sin(\theta + \pi))1_{(-\infty,0)}(r\cos(\theta + \pi)) \\ &= \frac{1}{2\pi}\exp(-r^2/2)r + \frac{1}{2\pi}\exp(-r^2/2)r \\ &= \frac{1}{\pi}\exp(-r^2/2)r \qquad \text{on} \ \ (0, \infty) \times (-\pi/2, \pi/2] \\ &= r\exp(-r^2/2) \cdot \frac{1}{\pi} = f_R(r)f_\Theta(\theta)\,. \end{aligned}$$

Thus $R$ and $\Theta$ are independent with densities $f_R(r) = r\exp(-r^2/2)1_{(0,\infty)}$ and $f_\Theta(\theta) = \pi^{-1}1_{(-\pi/2,\pi/2]}(\theta)$. Note that the distribution function of $R$ is given by

$$F_R(r) = \int_0^r f_R(y)dy = \int_0^r y\exp(-y^2/2)dy = 1 - \exp(-r^2/2)\,.$$

It follows easily from this that

$$F_{R^2}(x) = P(R^2 \le x) = P(R \le \sqrt{x}) = 1 - \exp(-x/2)$$

for $x \in [0, \infty)$; i.e. $R^2 \sim$ Exponential$(1/2) =$ Gamma$(1, 1/2) = \chi^2_2$.
(c) If $U$ and $V$ are independent Uniform$(0, 1)$ random variables, we can use the inverse transformation to first obtain

$$R^2 \equiv F^{-1}_{\chi^2_2}(U) = -2\log(1 - U) \sim \chi^2_2 \qquad \text{and} \qquad \Theta \equiv 2\pi V \sim \ \text{Uniform}(0, 2\pi)$$

note that $R^2$ and $\Theta$ are independent by independence of $U$ and $V$. Then in view of (b)

$$(X, Y) \equiv (R\cos(\Theta), R\sin(\Theta)) \sim N_2(0, I)\,.$$

6. Suppose that $U \sim$ Uniform$(0,1)$, $\alpha > 0$, and

$$X_n \equiv (n^\alpha / \log(n+1)) 1_{[0,1/n^\alpha]}(U) \, .$$

(a) Show that $X_n \rightarrow_{a.s.} 0$ and $E(X_n) \rightarrow E(0) = 0$.

(b) Can you find a random variable $Y$ with $|X_n| \le Y$ for all $n$ with $E(Y) < \infty$ for any $\alpha$?
(c) For what values of $\alpha$ does the uniform integrability condition

$$\limsup_{n \to \infty} E\{|X_n| 1_{[|X_n| \ge M]}\} \rightarrow 0 \qquad \text{as} \quad M \rightarrow \infty$$

hold?

**Solution:** (a) $X_n \rightarrow_{a.s.} 0$ since $X_n(\omega) = 0$ for $1/n^\alpha < U(\omega)$, or equivalently $n > (1/U(\omega))^{1/\alpha}$ and since $P(0 < U \le 1) = 1$. Moreover,

$$E(X_n) = \frac{n^\alpha}{\log(n+1)} \frac{1}{n^\alpha} = \frac{1}{\log(n+1)} \rightarrow 0 = E(0) \, .$$

(b) Here is the solution for the case $\alpha = 1$; in this case $n/\log(n+1)$ is monotone increasing in $n$. Then the smallest possible random variable $Y$ satisfying $|X_n| \le Y$ for all $n$ is $Y$ defined by

$$Y = \sum_{k=1}^\infty \frac{k}{\log(k+1)} 1_{(1/(k+1), 1/k]}(U) \, .$$

But we compute

$$\begin{aligned} E(Y) &= \sum_{k=1}^\infty \frac{k}{\log(k+1)} \left\{ \frac{1}{k} - \frac{1}{(k+1)} \right\} \\ &= \sum_{k=1}^\infty \frac{k}{\log(k+1)} \left\{ \frac{1}{k(k+1)} \right\} \\ &= \sum_{k=1}^\infty \frac{1}{(k+1)\log(k+1)} \\ &= \infty \, . \end{aligned}$$

Thus there is no integrable dominating function $Y$ for all the $X_n$'s if $\alpha = 1$. When $\alpha \geq 1/(2 \cdot \log 2) \approx .721\ldots$, the sequence $n^\alpha / \log(n+1)$ is again monotone, and the above method of proof applies with minor modifications. If $\alpha < 1/(2 \cdot \log(2))$, then the sequence $n^\alpha / \log(n+1)$ is not monotone increasing until $n \geq k_0(\alpha)$ defined by

$$\frac{k_0(\alpha)}{(k_0(\alpha)+1) \cdot \log(k_0(\lambda)+1)} < \alpha.$$

Then we claim only that $Y$ defined by

$$Y = \sum_{k=k_0(\alpha)}^{\infty} \frac{k^\alpha}{\log(k+1)} 1_{(1/(k+1),1/k]}(U)$$

dominates $X_n$ for $n \geq k_0(\alpha)$. A slight modification of the above proof shows that $E(Y) = \infty$ as well.

(c) On the other hand the uniform integrability condition does hold for any $\alpha > 0$:

$$\begin{aligned}
E\{|X_n| 1_{[|X_n| \geq M]}\} &= E\left\{\frac{n^\alpha}{\log(n+1)} 1_{[0,1/n^\alpha]}(U) 1\{(n^\alpha / \log(n+1)) \geq M, U \leq 1/n^\alpha\}\right\} \\
&= \frac{n^\alpha}{\log(n+1)} E\left\{1_{[0,1/n^\alpha]}(U)\right\} 1\{(n^\alpha / \log(n+1)) \geq M\} \\
&= \frac{1}{\log(n+1)} 1\{(n^\alpha / \log(n+1)) \geq M\} \\
&\to 0 \cdot 1 = 0
\end{aligned}$$

as $n \to \infty$ for every $\alpha > 0$ and $M > 0$. Hence the sequence $\{X_n\}$ is uniformly integrable for every $\alpha > 0$.