

Statistics 581, Problem Set 2 Solutions

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1. Suppose that Y is a random variable with $E(Y^2) < \infty$.

(a) Show that

$$\text{Var}(Y) = E\{\text{Var}(Y|X)\} + \text{Var}\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}.$$

(b) Interpret (a) geometrically.

(c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $\text{Var}(Y)$.

Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

Solution: (a) We compute directly:

$$\begin{aligned} \text{Var}(Y) &= E[Y - E(Y)]^2 = E[Y - E(Y|X) + E(Y|X) - E(Y)]^2 \\ &= E[Y - E(Y|X)]^2 + 2E[(Y - E(Y|X))[E(Y|X) - E(Y)]] \\ &\quad + E[E(Y|X) - E(Y)]^2 \\ &= E\{E\{[Y - E(Y|X)]^2|X\}\} + 0 + \text{Var}[E(Y|X)] \\ &= E\{\text{Var}[Y|X]\} + \text{Var}[E(Y|X)] \end{aligned}$$

since, by computing conditionally,

$$\begin{aligned} E[(Y - E(Y|X))[E(Y|X) - E(Y)]] &= E\{E\{[(Y - E(Y|X))[E(Y|X) - E(Y)]|X\}\} \\ &= E\{[E(Y|X) - E(Y)]E\{[Y - E(Y|X)]|X\}\} \\ &= E\{[E(Y|X) - E(Y)]\{E(Y|X) - E(Y|X)\}\} \\ &= E\{[E(Y|X) - E(Y)] \cdot 0\} \\ &= 0. \end{aligned}$$

(b) A geometric interpretation of (a) is that $Y - E(Y|X)$ is orthogonal to $E(Y|X) - E(Y)$ in $L_2(\Omega, \mathcal{A}, P) = L_2(P)$, thus the identity in (a) can be interpreted as a “pythagorean theorem”. Also note that $Y - E(Y|X)$ is orthogonal to any function $g(X)$: much as in the last part of (a)

$$\begin{aligned} E\{[(Y - E(Y|X))]g(X)\} &= E\{E\{[(Y - E(Y|X))]g(X)|X\}\} \\ &= E\{g(X)E\{[Y - E(Y|X)]|X\}\} \\ &= E\{g(X)\{E(Y|X) - E(Y|X)\}\} \\ &= E\{g(X) \cdot 0\} \\ &= 0. \end{aligned}$$

(c) Now $(Y|K) \sim \chi_{2K+n}^2$ where $K \sim \text{Poisson}(\delta/2)$, so

$$E(Y) = E\{E(Y|K)\} = E\{2K + n\} = n + 2(\delta/2) = n + \delta.$$

Furthermore, using part (a) we get

$$\begin{aligned} \text{Var}(Y) &= E\{\text{Var}(Y|K)\} + \text{Var}\{E(Y|K)\} \\ &= E\{2(2K + n)\} + \text{Var}\{2K + n\} \\ &= 4(\delta/2) + 2n + 4(\delta/2) \\ &= 2n + 4\delta. \end{aligned}$$

2. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

Solution: See Ferguson, ACILST, page 63 (and page 57). Find Γ orthogonal so that $\Gamma'\Sigma\Gamma = D$ where D is diagonal. Now $\Gamma\Gamma' = I$, so if $\Sigma^2 = \Sigma$ we have $D^2 = \Gamma'\Sigma\Gamma\Gamma'\Sigma\Gamma = \Gamma'\Sigma^2\Gamma = \Gamma'\Sigma\Gamma = D$ and conversely. Moreover, since Σ is of rank k , D is of rank k , and this together with $D^2 = D$ implies that D has k 1's on the diagonal and $n-k$ 0's. Without loss, assume that Γ has been chosen so that the k ones occur in the the first r positions of the diagonal matrix D ; thus

$$D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is $k \times k$. Moreover, note that

$$\begin{aligned} D\Gamma'\mu &= \Gamma'\Sigma\Gamma\Gamma'\mu = \Gamma'\Sigma\mu \\ &= \Gamma'\mu, \end{aligned}$$

and this implies that the last $n - k$ components of $\Gamma'\mu$ are all zero. Now let $Y = \Gamma'X$ (much as in the proof of theorem 1.3.2 of the notes). Then $Y \sim N_n(\Gamma'\mu, D)$, $Y'Y = X'\Gamma\Gamma'X = X'X$, and by (d) of page 16, section 1.3, $X'X = Y'Y \sim \chi_k^2(\delta)$ where $\delta = (\Gamma'\mu)'(\Gamma'\mu) = \mu'\Gamma\Gamma'\mu$.

3. Ferguson, ACILST, #1, page 11.

Let X_1, X_2, \dots be i.i.d. random variables with densities $f(x) = \alpha x^{-(\alpha+1)} 1_{(1, \infty)}(x)$.

- (a) For what values of $\alpha > 0$ and $r > 0$ is it true that $n^{-1}X_n \rightarrow_r 0$?
 (b) For what values of $\alpha > 0$ is it true that $n^{-1}X_n \rightarrow_{a.s.} 0$?
 (c) If X_1, X_2, \dots are independent with X_n having density $f_n(x) = \alpha_n x^{-(\alpha_n+1)} 1_{(1, \infty)}(x)$ for $n = 1, 2, \dots$, Find the limit of $n^{-2}EX_n^2$ when $\alpha_n = 2 + n^{-\gamma}$ for $\gamma \in \mathbb{R}$.

Solution: (a) $E(X_n^r) = \alpha \int_1^\infty x^r x^{-(\alpha+1)} dx = \alpha/(\alpha - r)$ if $\alpha > r$, while $E(X_n^r) = \infty$ if $\alpha \leq r$. Thus $E[(n^{-1}X_n)^r] = n^{-r}(\alpha/(\alpha - r)) \rightarrow 0$ if $\alpha > r$.

(b) $P(n^{-1}X_n > \epsilon) = P(X_n > n\epsilon) = \alpha \int_{n\epsilon}^\infty x^{-(\alpha+1)} dx = (n\epsilon)^{-\alpha}$, so for $\alpha > 1$ we have

$$\sum_{n=1}^{\infty} P(n^{-1}X_n > \epsilon) \leq \sum_{n=1}^{\infty} (n\epsilon)^{-\alpha} < \infty.$$

Thus $P(n^{-1}X_n > \epsilon \text{ i.o.}) = 0$ by the Borel-Cantelli lemma, and we conclude that $n^{-1}X_n \rightarrow_{a.s.} 0$ for $\alpha > 1$. [Since the X_n 's are independent, $n^{-1}X_n \rightarrow_{a.s.} 0$ if and only if $\alpha > 1$ by the converse Borel-Cantelli lemma.]

(c) If the X_n 's are independent with the same densities as in (a) and (b) but with $\alpha = \alpha_n = 2 + n^{-\gamma}$ for X_n , then $EX_n^2 = (2 + n^{-\gamma})/n^{-\gamma}$, so

$$n^{-2}EX_n^2 = (2 + n^{-\gamma})/n^{2-\gamma} \rightarrow \begin{cases} 0, & \text{if } \gamma < 2, \\ 2, & \text{if } \gamma = 2, \\ \infty, & \text{if } \gamma > 2. \end{cases}$$

4. (a) Ferguson, ACILST, #4, page 6: Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow 1$.
 (b) Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow \infty$.
 (c) Give an example of a sequence of random variables X_n for which $X_n \rightarrow_p 0$ but $X_n \rightarrow_{a.s.} 0$ fails.

Solution: (a) If $X_n = a_n$ with probability p_n and $X_n = 0$ with probability $1 - p_n$, then $E(X_n) = a_n p_n$ and $E(X_n^2) = a_n^2 p_n = 1$ if $p_n = 1/a_n^2$. Then $E(X_n) = a_n/a_n^2 = 1/a_n \rightarrow 0$ if $a_n \rightarrow \infty$. Ferguson's solution on

page 173 takes $a_n = n$; the same holds for any sequence $a_n \rightarrow \infty$.

(b) Let $U \sim \text{Uniform}(0, 1)$, and set $X_n = n^\alpha 1_{[0, 1/n]}(U)$. Then $EX_n = n^\alpha n^{-1} \rightarrow 0$ if $\alpha < 1$, while $EX_n^2 = n^{2\alpha} n^{-1} \rightarrow \infty$ if $\alpha > 1/2$. Thus the required convergences hold for all $1/2 < \alpha < 1$.

(c) Let $U \sim \text{Uniform}(0, 1)$. The “dancing functions” are defined by $X_{n,k} = 1_{[(k-1)/2^n, k/2^n)}(U)$, $k = 1, \dots, 2^n$, $n = 1, 2, \dots$. Let $\{Y_m\}_{m \geq 1}$ be defined by $Y_m = X_{n,k}$ if $m = (\sum_{j=1}^n 2^j) + k = 2^{n+1} - 2 + k$ with $1 \leq k \leq 2^n$. Then for $\epsilon \in (0, 1)$,

$$P(|Y_m| > \epsilon) = P(|X_{n,k}| > \epsilon) = 2^{-n} \rightarrow 0$$

so $Y_m \rightarrow_p 0$, but for every $U(\omega) \in (0, 1)$ we have $Y_m(\omega) = 1$ for infinitely many m 's and also $Y_m(\omega) = 0$ for infinitely many m 's. Hence

$$0 = \liminf Y_m < \limsup Y_m = 1 \quad a.s.$$

and it follows that Y_m does not converge to 0 almost surely.

5. (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W , distribution function F_W , and inverse distribution function F_W^{-1} explicitly.
 (b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(0, 2\pi)$.
 (c) Use the results of (a) and (b) to show (using Theorem 2.3.1, Chapter 2 notes, page 13) how to use two independent $\text{Uniform}(0, 1)$ random variables U and V to generate two standard normal random variables.

Solution: (a) If $W \sim \chi_2^2 = \text{Gamma}(1, 1/2)$, the density function is given by $f_W(w) = (1/2)e^{-w/2} 1_{[0, \infty)}$; i.e. $W \sim \text{Exponential}(1/2)$. Hence the distribution function is $F_W(w) = 1 - \exp(-w/2)$ for $w \geq 0$, and the inverse distribution function is $F_W^{-1}(u) = -2 \log(1 - u)$.

(b) The joint density of (X, Y) is given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi} \exp(-(x^2 + y^2)/2) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Moreover, $X = R \cos(\Theta)$ and $Y = R \sin(\Theta)$. Hence the Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \left| \begin{pmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{pmatrix} \right| = r \cos^2(\theta) + r \sin^2(\theta) = r.$$

Thus we find that the joint density of (R, Θ) is given by

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= f_{X,Y}(r \cos(\theta), r \sin(\theta)) = \frac{1}{2\pi} \exp(-r^2/2)r \quad \text{on } (0, \infty) \times [0, 2\pi] \\ &= r \exp(-r^2/2) \cdot \frac{1}{2\pi} = f_R(r)f_\Theta(\theta). \end{aligned}$$

Thus R and Θ are independent with densities $f_R(r) = r \exp(-r^2/2)1_{(0,\infty)}$ and $f_\Theta(\theta) = (2\pi)^{-1}1_{[0,2\pi]}(\theta)$. Note that the distribution function of R is given by

$$F_R(r) = \int_0^r f_R(y)dy = \int_0^r y \exp(-y^2/2)dy = 1 - \exp(-r^2/2).$$

It follows easily from this that

$$F_{R^2}(x) = P(R^2 \leq x) = P(R \leq \sqrt{x}) = 1 - \exp(-x/2)$$

for $x \in [0, \infty)$; i.e. $R^2 \sim \text{Exponential}(1/2) = \text{Gamma}(1, 1/2) = \chi_2^2$.

(c) If U and V are independent $\text{Uniform}(0, 1)$ random variables, we can use the inverse transformation to first obtain

$$R^2 \equiv F_{\chi_2^2}^{-1}(U) = -2 \log(1-U) \sim \chi_2^2 \quad \text{and} \quad \Theta \equiv 2\pi V \sim \text{Uniform}(0, 2\pi)$$

note that R^2 and Θ are independent by independence of U and V . Then in view of (b)

$$(X, Y) \equiv (R \cos(\Theta), R \sin(\Theta)) \sim N_2(0, I).$$

6. Suppose that $U \sim \text{Uniform}(0, 1)$, $\alpha > 0$, and

$$X_n \equiv (n^\alpha / \log(n+1))1_{[0, 1/n^\alpha]}(U).$$

(a) Show that $X_n \rightarrow_{a.s.} 0$ and $E(X_n) \rightarrow E(0) = 0$.

(b) Can you find a random variable Y with $|X_n| \leq Y$ for all n with $E(Y) < \infty$ for any α ?

(c) For what values of α does the uniform integrability condition

$$\limsup_{n \rightarrow \infty} E\{|X_n|1_{\{|X_n| \geq M\}}\} \rightarrow 0 \quad \text{as } M \rightarrow \infty$$

hold?

Solution: (a) $X_n \rightarrow_{a.s.} 0$ since $X_n(\omega) = 0$ for $1/n^\alpha < U(\omega)$, or equivalently $n > (1/U(\omega))^{1/\alpha}$ and since $P(0 < U \leq 1) = 1$. Moreover,

$$E(X_n) = \frac{n^\alpha}{\log(n+1)} \frac{1}{n^\alpha} = \frac{1}{\log(n+1)} \rightarrow 0 = E(0).$$

(b) Now the smallest possible random variable Y satisfying $|X_n| \leq Y$ for all n is Y defined by

$$Y = \sum_{k=1}^{\infty} \frac{k^\alpha}{\log(k+1)} 1_{(1/(k+1)^\alpha, 1/k^\alpha]}(U).$$

But we compute

$$\begin{aligned} E(Y) &= \sum_{k=1}^{\infty} \frac{k^\alpha}{\log(k+1)} \left\{ \frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} \right\} \\ &= \sum_{k=1}^{\infty} \frac{1}{\log(k+1)} \left\{ 1 - \left(\frac{k}{k+1} \right)^\alpha \right\} \\ &= \sum_{k=1}^{\infty} \frac{1}{\log(k+1)} \left\{ 1 - \left(1 - \frac{1}{k+1} \right)^\alpha \right\} \\ &\geq \sum_{k=1}^{k(\alpha)} \frac{1}{\log(k+1)} \left\{ 1 - \left(1 - \frac{1}{k+1} \right)^\alpha \right\} \\ &\quad + \sum_{k=k(\alpha)}^{\infty} \frac{1}{\log(k+1)} \frac{\alpha/2}{k+1} \\ &= \infty. \end{aligned}$$

since $(1-x)^\alpha \leq 1 - \alpha x/2$ for $x \leq x(\alpha)$. Thus there is no integrable dominating function Y for any value of α .

(c) On the other hand the uniform integrability condition does hold for any $\alpha > 0$:

$$\begin{aligned} E\{|X_n| 1_{\{|X_n| \geq M\}}\} &= E \left\{ \frac{n^\alpha}{\log(n+1)} 1_{[0, 1/n^\alpha]}(U) 1_{\{(n^\alpha / \log(n+1)) \geq M, U \leq 1/n^\alpha\}} \right\} \\ &= \frac{n^\alpha}{\log(n+1)} E \left\{ 1_{[0, 1/n^\alpha]}(U) \right\} 1_{\{(n^\alpha / \log(n+1)) \geq M\}} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\log(n+1)} 1_{\{(n^\alpha/\log(n+1)) \geq M\}} \\ &\rightarrow 0 \cdot 1 = 0 \end{aligned}$$

as $n \rightarrow \infty$ for every $\alpha > 0$ and $M > 0$. Hence the sequence $\{X_n\}$ is uniformly integrable for every $\alpha > 0$.