Statistics 581, Problem Set 8

Wellner; 11/18/2009

Reading: Chapter 3, Sections 3-5;
Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;
Lehmann and Casella, Chapter 6, especially sections 6.1-6.2, pages 429-443.
Due: Wednesday, November 25, 2009.

1. Suppose that $\theta = (\theta_1, \theta_2) \in \Theta \subset R^k$ where $\theta_1 \in R$ and $\theta_2 \in R^{k-1}$. Show that:
A. $\mathbf{l}_1^* = \dot{\mathbf{l}}_1 - I_{12} I_{22}^{-1} \dot{\mathbf{l}}_2$ is orthogonal to $[\dot{\mathbf{l}}_2] \equiv \{a' \dot{\mathbf{l}}_2 : a \in R^{k-1}\}$ in $L_2(P_\theta)$.
B. $I_{11\cdot 2} = \inf_{c \in R^{k-1}} E_\theta(\dot{\mathbf{l}}_1 - c' \dot{\mathbf{l}}_2)^2$ and that the minimum is achieved when $c' = I_{12} I_{22}^{-1}$. Thus
$$I_{11\cdot 2} = E_\theta(\dot{\mathbf{l}}_1 - I_{12} I_{22}^{-1} \dot{\mathbf{l}}_2)^2 = E_\theta[(\mathbf{l}_\theta^*)^2] \, .$$
C. Prove the formulas (16) and (17) on page 21 of the Chapter 3 notes and interpret these formulas geometrically.

2. Suppose that $(Y|Z) \sim \text{Poisson}(\lambda e^{\gamma Z})$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ. You may assume that
$$a(z) \equiv (\partial / \partial \eta) \log g_\eta(z)$$
exists and $E\{a^2(Z)\} < \infty$. Thus Z is a "covariate" or "predictor variable", γ is a "regression parameter" which affects the intensity of the (conditionally) Poisson variable Y, and $\theta = (\gamma, \lambda, \eta)$.
(a) Find the information matrix for θ. What does the structure of this matrix say about the effect of η being known or unknown about the estimation of γ and λ?
(b) Find the information and information bound for γ if the parameter λ is known.
(c) Find the efficient score function and the efficient influence function for estimation of γ when λ is known.
(d) Find the information and information bound for γ if the parameter λ is unknown, $I_{\gamma\gamma\cdot\lambda}$.
(e) Find the efficient score function and the efficient influence function for estimation of γ when λ is unknown.
(f) In the case when $Z \sim \text{Bernoulli}(\eta)$, compute the ratio of the information for γ when λ is unknown, to the information for γ when λ is known as a function of γ and of η.

3. (a) Lehmann and Casella, Problem 2.13, page 501.
(b) Let $R_n(\theta) \equiv n E_\theta (T_n - \theta)^2$ where T_n is the Hodges superefficient estimator as in Example 3.3.1 (so $T_n = \delta_n$ of Example 2.5, Lehmann and Casella pages 440 - 443). Show that $R_n(n^{-1/4}) \to \infty$ as $n \to \infty$.

4. Suppose that $Z \sim N(0,1)$ and, for $\mu \in R$ and $\sigma > 0$, that $X = \mu + \sigma Z \sim P_{\mu,\sigma} = N(\mu, \sigma^2)$.
(a) Compute the likelihood ratio
$$\frac{dP_{\mu,\sigma}}{dP_{0,\sigma}}(x) = \frac{\sigma^{-1}\phi((x-\mu)/\sigma)}{\sigma^{-1}\phi(x/\sigma)} \qquad \text{and} \qquad Y \equiv \log \frac{dP_{\mu,\sigma}}{dP_{0,\sigma}}(X) \, .$$

What is the distribution of Y under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$?
(b) Plot the function $l(\mu; X) \equiv \log(dP_{\mu,\sigma}/dP_{0,\sigma})(X)$ as a function of μ.
(c) Find the maximum value of the function $l(\mu; X)$ in B (as a function of μ) and the value of $\mu \equiv \hat{\mu}$ which achieves the maximum.
(d) What is the distribution of $\hat{\mu}$ under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$? What is the distribution of $l(\hat{\mu}; X)$ under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$?

5. Read Note 8.5, Lehmann and Casella, page 145. Explore the identity in the second display in this note and see if it makes sense as written. If not, rewrite the identity in a way that makes sense to you. [Compare with Efron and Johnstone (1990) and/or Bickel, Klaassen, Ritov, and Wellner (1993), pages 420-424.]

6. **Optional bonus problem:** Suppose that $X_1, \ldots, X_n$ are i.i.d. F on $\mathbb{R}$, and let $\mathbb{F}_n$ denote the empirical d.f. of the X_i's. Let Φ denote the standard normal distribution function, $\Phi(x) = \int_{-\infty}^{x} \phi(y)dy$ where $\phi(y) = (2\pi)^{-1/2}\exp(-y^2/2)$ is the standard normal density. Let $0 < a < 1$ and define a new estimator $\widetilde{F}_n$ of F by

$$\widetilde{F}_n(x) = \begin{cases} (1-a)\Phi(x) + a\mathbb{F}_n(x), & \text{if } \|\mathbb{F}_n - \Phi\|_\infty \le n^{-1/4}, \\ \mathbb{F}_n(x), & \text{if } \|\mathbb{F}_n - \Phi\|_\infty > n^{-1/4}. \end{cases}$$

(a) Find the limiting distribution of the process $\{\sqrt{n}(\widetilde{F}_n(x) - F(x)) : x \in \mathbb{R}\}$ when $F = \Phi$.
(b) Find the limiting distribution of the process $\{\sqrt{n}(\widetilde{F}_n(x) - F(x)) : x \in \mathbb{R}\}$ when $F \neq \Phi$.
(c) Show that $\widetilde{F}_n$ is not a regular estimator of F at $F = \Phi$ (in an appropriate sense to be defined), but that F is a regular estimator of F at any $F \neq \Phi$.