Statistics 581, Problem Set 2

Wellner; 10/7/2009

Reading: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 1-25.
Due: Wednesday, October 14, 2009.

1. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that
$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\};$$
i.e.
$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}.$$
(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

2. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

3. Ferguson, ACILST, #1, page 11.
Let $X_1, X_2, \ldots$ be i.i.d. random variables with densities $f(x) = \alpha x^{-(\alpha+1)} 1_{(1,\infty)}(x)$.
(a) For what values of $\alpha > 0$ and $r > 0$ is it true that $n^{-1}X_n \rightarrow_r 0$?
(b) For what values of $\alpha > 0$ is it true that $n^{-1}X_n \rightarrow_{a.s.} 0$?
(c) If $X_1, X_2, \ldots$ are independent with X_n having density $f_n(x) = \alpha_n x^{-(\alpha_n+1)} 1_{(1,\infty)}(x)$ for $n = 1, 2, \ldots$, Find the limit of $n^{-2}EX_n^2$ when $\alpha_n = 2 + n^{-\gamma}$ for $\gamma \in \mathbb{R}$.

4. (a) Ferguson, ACILST, #4, page 6: Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow 1$.
(b) Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow \infty$.
(c) Give an example of a sequence of random variables X_n for which $X_n \rightarrow_p 0$ but $X_n \rightarrow_{a.s.} 0$ fails.

5. (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W, distribution function F_W, and inverse distribution function F_W^{-1} explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by

$R^2 = X^2+Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(0, 2\pi)$.
(c) Use the results of (a) and (b) to show (using Theorem 2.3.1, Chapter 2 notes, page 13) how to use two independent Uniform$(0,1)$ random variables U and V to generate two standard normal random variables.

6. Suppose that $U \sim \text{Uniform}(0,1)$, $\alpha > 0$, and

$$X_n \equiv (n^\alpha / \log(n+1))1_{[0,1/n^\alpha]}(U)\,.$$

(a) Show that $X_n \rightarrow_{a.s.} 0$ and $E(X_n) \rightarrow E(0) = 0$.

(b) Can you find a random variable Y with $|X_n| \le Y$ for all n with $E(Y) < \infty$ for any α?
(c) For what values of α does the uniform integrability condition

$$\limsup_{n\to\infty} E\{|X_n|1_{[|X_n|\ge M]}\} \to 0 \qquad \text{as} \quad M \to \infty$$

hold?

7. **Optional Bonus Problem 1:**
(a) Lehmann and Casella, #1.7, page 62.
(b) Lehmann and Casella, #1.8, page 62.
(c) Lehmann and Casella, #1.9, page 62.

8. **Optional Bonus Problem 2:** Lehmann and Casella, #1.10, page 62.