Statistics 581, Problem Set 10

Wellner; 12/2/2009

Reading: Chapter 4, Sections 1-4;
Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;
Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.
Due: Wednesday, December 9, 2009.

1. Suppose that (as in Lemma 5.2, page 38, Chapter 3 Notes) P and Q are two probability measures on a measurable space $(\mathcal{X}, \mathcal{A})$ with densities p and q with respect to a $\sigma-$finite dominating measure μ, and P^n and Q^n denote the corresponding product measures on $(\mathcal{X}^n, \mathcal{A}_n)$ (of $X_1, \ldots, X_n$ i.i.d. as P or Q respectively).
(a) What is the relationship between $K(P^n, Q^n)$ and $K(P, Q)$, if any?
(b) If P is the Normal$(0, \sigma^2)$ distribution and Q is the Normal(μ, σ^2) distribution, compute $K(P, Q)$, $\rho(P, Q) = \int \sqrt{pq}d\mu$, and $H^2(P, Q)$.
(c) Use the results of (a) and (b) together with Lemma 5.2 to calculate $K(P^n, Q^n)$, $\rho(P^n, Q^n)$, and $H^2(P^n, Q^n)$ when P and Q are as in (b).
(d) Find a sequence μ_n so that, with Q_n being the Normal distribution with mean μ_n, the quantities $K(P^n, Q_n^n)$, $\rho(P^n, Q_n^n)$, and $H^2(P^n, Q_n^n)$ converge to finite limits as $n \to \infty$.

2. Consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities
$$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1} \exp(-(\frac{x}{\alpha})^\beta) 1_{[0,\infty)}(x)$$
where $\theta \equiv (\alpha, \beta) \in (0, \infty) \times (0, \infty) \subset R^2$. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with density function p_θ.
(a) If $X \sim P_\theta \in \mathcal{P}$, show that the distributions of $-\log X$ form a location and scale family from a Gumbel (extreme value) density on R.
(b) Use the result of (a) to construct method of moments estimators or quantile based estimators $\overline{\theta}_n$ of $\theta = (\alpha, \beta)$.
(c) Show that the method of moments or quantile estimators $\overline{\theta}_n$ of θ are asymptotically normal, and find the asymptotic distribution; i.e. show that
$$\sqrt{n}(\overline{\theta}_n - \theta) \to_d N_2(0, \Sigma) \qquad \text{for some} \qquad \Sigma.$$
[We will use these estimators as "starting points" approximate (or one-step) maximum likelihood estimators in the next problem .]

3. (Problem #2, continued).
(a) Does a maximum likelihood estimate of $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ exist? Is it unique? (See Lehmann and Casella, Example 6.1, page 468.)
(b) Compute an approximate (one - step) maximum likelihood estimate $\check{\theta}$ of θ using

the method of moment (or quantile) estimators $\overline{\theta}_n$ as the preliminary estimators based on the following data (with $n = 12$):

$$1,\ 1.3,\ 1.7,\ 3.2,\ 10.7,\ 24.3,\ 51.2,\ 77.1,\ 93.7,\ 105,\ 111,\ 305.$$

[These are failure times in seconds for "breakdown" of an insulating fluid between two electrodes subject to a voltage of 40 kV. – from Nelson, *Applied Life Data Analysis*, page 252, but with some modifications or "recording errors".]
(c) Compute the maximum likelihood estimator $\hat{\theta}_n$, and compare it with the one step estimator computed in E.

4. Suppose that we want to model the survival of twins with a common genetic defect, but with one of the two twins receiving some treatment. Let X represent the survival time of the untreated twin and let Y represent the survival time of the treated twin. One (overly simple) preliminary model might be to assume that X and Y are independent with Exponential(η) and Exponential($\nu\eta$) distributions, respectively:

$$f_{\nu,\eta}(x,y) = \eta e^{-\eta x} \eta\nu e^{-\eta\nu y} 1_{(0,\infty)}(x) 1_{(0,\infty)}(y)$$

A. One crude approach to estimation in this problem is to reduce the data to $W = X/Y$, the maximal invariant for the group of scale changes $g(x,y) = (cx, cy)$ with $c > 0$. Find the distribution of W, and compute the Cramér-Rao lower bound for unbiased estimates of ν based on $W_1, \ldots, W_n$ with $W_i = X_i/Y_i$ and (X_i, Y_i) i.i.d. as (X, Y).
B. Find the information bound for estimation of ν based on observation of (X, Y) pairs when η is known and unknown.
C. Compare the bounds you computed in A and B and discuss the pros and cons of reducing to estimation based on the ratio $W = X/Y$.
D. Find the MLE $\hat{\nu}_n$ of ν based on (X_i, Y_i), $i = 1, \ldots, n$, and the MLE $\hat{\nu}_n^{(r)}$ of ν based on the reduced data $W_1, \ldots, W_n$. What are the limiting distributions of $\sqrt{n}(\hat{\nu}_n - \nu)$ and $\sqrt{n}(\hat{\nu}_n^{(r)} - \nu)$?

5. **Optional bonus problem:**
(a) Ferguson, ACLST, page 139, problem 3.
(b) What if Ferguson's density $f(x|\theta)$ with $\theta \in (0,1)$ is replaced by $\theta = (\gamma, \eta) \in (0,1) \times (0,\infty)$ and

$$f(x|\theta) \equiv f(x|\gamma,\eta) = \{(1-\gamma)e^{-x} + \gamma\eta^2 x \exp(-\eta x)\} 1_{[0,\infty)}(x)?$$

Can you estimate γ and η by the method of moments? Can you improve method of moment estimators via one-step estimators?