Statistics 581, Problem Set 1

Wellner; 9/30/2009

Reading: Lehmann & Casella, TPE, pages 1 - 32; Ferguson, ACILST, Chapter 1, pages 3-7; skim Chapter 0 handout; read Chapter 1 handout.
Due: Wednesday, October 7, 2009.

1. Let X and Y be i.i.d. Uniform$(0,1)$ random variables Define $U = X+Y$, $V = \min(X,Y) = X \wedge Y$.

   (i) What is the range of (U,V)?

   (ii) Find the joint density function $f_{U,V}(u,v)$ of the pair (U,V). Are U and V independent?

2. (a) Ferguson, ACILST, #2, page 6.
   (b) Now suppose that $U \sim$ Uniform$(0,1)$ and for each $n \geq 1$ define $V_n \equiv \sum_{j=1}^{n} (j/n) 1_{((j-1)/n, j/n]}(U)$. Show that $V_n \stackrel{d}{=} X_n$ where X_n is as in (a).
   (c) Show that $V_n \to_p U$.

3. Lehmann & Casella, TPE, problem 5.33, page 69.
   Morris (1982, 1983b) investigated the properties of natural exponential families with quadratic variance functions. There are only six such families: normal, binomial, gamma, Poisson, negative binomial, and the lesser-known generalized hyperbolic secant distribution, which is the density of $X = \pi^{-1}\log(Y/(1-Y))$ when $Y \sim$ Beta$((1/2)+\theta/\pi, (1/2)-\theta/\pi)$ $|\theta| < \pi/2$. For this sixth family:
   (a) Find the density of X and show that it constitutes an exponential family.
   (b) Find the mean and variance of X, and show that the variance equals $1+\mu^2$ where μ is the mean.
   (Subsequent work on quadratic and other power variance families has been done by Bar-Lev and Enis (1986, 1988), Bar-Lev and Bshouty (1989), and Letac and Mora (1990).)

4. (a) Lehmann & Casella, TPE, problem 3.5, page 64.
   Let S be the support of a distribution on a Euclidean space $(\mathcal{X}, \mathcal{A})$;

i.e. the set of all points x for which $P(A) > 0$ for all open rectangles $A = \{(x_1, \ldots, x_n) : \ a_i < x_i < b_i, \ i = 1, \ldots, n\}$ for numbers $a_i < b_i$ in R.) Show that: (i) S is closed; (ii) $P(S) = 1$; (iii) S is the intersection of all closed sets C with $P(C) = 1$.
(b) Lehmann & Casella, TPE, problem 3.6, page 64.
Show that if P and Q are two probability measures over the same Euclidean space which are equivalent, then they have the same support.
(c) Lehmann & Casella, TPE, problem 3.7, page 64.
Let P and Q assign probabilities

$$P: \quad P(X = 1/n) = p_n > 0, \quad n = 1, 2, \ldots \quad (\sum_n p_n = 1),$$

$$Q: \quad P(X = 0) = 1/2; \quad P(X = 1/n) = q_n > 0, \quad n = 1, 2, \ldots (\sum_n q_n = 1/2).$$

Then, show that P and Q have the same support but are not equivalent.

5. (a) Lehmann and Casella, TPE, problem 1.2, page 62.
Let $X_1, \ldots, X_n$ be uncorrelated random variables with common expectation θ and variance σ^2. Show that among all linear estimators $\sum_1^n \alpha_i X_i$ of θ satisfying $\sum_1^n \alpha_i = 1$, the mean $\overline{X}_n$ has the smallest variance.
(b) Lehmann and Casella, TPE, problem 1.3, page 62.
In the preceding problem minimize the variance of $\sum_1^n \alpha_i X_i$ over α_i's with $\sum_1^n \alpha_i = 1$ when:
(a') The variance of X_i is σ^2/c_i, $c_i > 0$ and known.
(b') The X_i have common variance σ^2 but are correlated with common correlation coefficient ρ.

6. **Bonus problem 1:** Suppose that $X \sim$ Uniform$(0, 1)$ and $Y = -\log(1 - X)$. Find the joint distribution function $F(x, y) = F_{X,Y}(x, y)$ of (X, Y). Does this distribution function have a joint density with respect to Lebesgue measure?

7. **Bonus problem 2:** Lehmann and Casella, TPE, problem 3.8, page 64, modified as follows: suppose that X and Y are independent random variables with $X \sim E(0, \lambda)$ and $Y \sim E(0, \mu)$.