Statistics 581, Final Exam
Wellner; 12/8/2008

This exam is to be taken without the use of any books or notes.

1. (48) points) **Define** *each* of the following terms. In each case, provide an appropriate context for your definition.
   (a) An asymptotically linear estimator T_n of a parameter $\nu(P)$ with influence function ψ.
   (b) A locally regular estimator T_n of a parameter $\nu(P_\theta)$.
   (c) An $n^{1/4}-$consistent *preliminary estimator* of a (vector) parameter θ in a parametric model.
   (d) A *one-step estimator* of a (vector) parameter θ in a parametric model.

2. (32 points) **State** *four* of the following six results, providing the appropriate (brief) context for your statement:
   (a) The Mann-Wald (or continuous mapping) theorem.
   (b) The basic event identity connected with the inverse transformation.
   (c) A result concerning the Kullback-Leibler discrepancy $K(P,Q)$ for two probability measures P and Q.
   (d) An identity which gives two ways of calculating the information matrix; specify the regularity conditions which are needed for the identity to hold.
   (e) The Lindeberg-Feller central limit theorem.
   (f) LAN (Local Asymptotic Normality) of the local - log likelihood ratios for a regular parametric model satisfying the Cramér hypotheses).

Do either problem 3 or problem 4.

3. (40 points) Let $X_1, \ldots, X_n$ be i.i.d. $P_\theta = \text{Normal}(\theta, 1)$.
   (a) Give the Hodges superefficient estimator T_n of θ (with superefficiency at $\theta = 0$).
   (b) What is the limiting distribution of $\sqrt{n}(T_n - \theta)$ as a function of θ?
   (c) What is the limiting distribution of $\sqrt{n}(T_n - \theta_n)$ when sampling from $\theta = \theta_n$ when $\theta_n = cn^{-1/2}$?
   (d) Does the limit distribution in (c) depend on c? Is T_n a locally regular estimator of θ at $\theta = 0$?
   (e) What is the limit of $E_{\theta_n}\{[\sqrt{n}(T_n - \theta_n)]^2\}$ when $\theta_n = cn^{-1/2}$ as in (c)? For what values of c does the limiting risk of T_n exceed the (limiting) risk of $\overline{X}_n$?

4. (40 points)
   (a) Give an example of a parametric model for which the likelihood equations

may have multiple roots, but for which our general theorem proving existence of consistent and asymptotically efficient roots applies.
(b) Give an example of a parametric model for which the maximum likelihood estimator is inconsistent.
(c) Give an example of a parametric model for which the maximum likelihood estimator does not exist.
(d) Give an example of a (non-Gaussian, non-normal) regular parametric model with dimension d of the parameter space at least 2 for which the maximum likelihood estimator exists almost surely and is asymptotically efficient. (e) Give an example of a one-dimensional parametric model for which the second derivative identity for information fails to hold.

5. (30 points)
(a) **State** the Glivenko-Cantelli theorem. Then **prove** that it holds if it holds for the case of i.i.d. Uniform$(0,1)$ random variables.
(b) **Prove** the Glivenko-Cantelli theorem for i.i.d. Uniform$(0,1)$ random variables: if $\xi_1, \ldots, \xi_n, \ldots$ are i.i.d. Uniform$(0,1)$ with empirical distribution function

$$\mathbb{G}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{[0,t]}(\xi_i)\,, \qquad \text{then} \qquad \sup_{0 \le t \le 1} |\mathbb{G}_n(t) - t| \to_{a.s.} 0\,.$$

6. (50 points) (A parametric version of the Cox model.)
Suppose that $(Y|Z) \sim \text{Exponential}(\lambda e^{\gamma Z})$ where $Z \sim \text{Bernoulli}(\eta)$. Thus the density of $X = (Y, Z)$ is given by

$$p_\theta(y, z) = \lambda e^{\gamma z} \exp(-\lambda e^{\gamma z} y) 1_{[0,\infty)}(y) \eta^z (1-\eta)^{1-z} 1_{\{0,1\}}(z)$$

where $\theta = (\gamma, \lambda, \eta)$. Suppose that $X_1 = (Y_1, Z_1), \ldots, X_n = (Y_n, Z_n)$ are i.i.d. as X.
(a) Find the scores for $\theta = (\gamma, \lambda, \eta)$ based on one observation.
(b) Find the information matrix for θ.
(c) Compute the information for γ when λ is known (I_{11}) and unknown ($I_{11 \cdot 2}$), and explain the difference based on the geometry of the scores.
(d) Write down the score equations for θ and briefly discuss the existence and uniqueness of solutions of these equations.
(e) What does our theory from chapter 4 say about the limiting distribution of $\sqrt{n}(\widehat{\theta} - \theta_0)$ and of $\sqrt{n}(\widehat{\gamma} - \gamma_0)$?

7. (60 points). (Parametric Cox model, continued).
(a) Suggest three tests of the (composite!) null hypothesis $H : \gamma = 0$ versus $K : \gamma \neq 0$. What is the asymptotic distribution of each of these three statistics under the null hypothesis and under local alternatives of the form $\gamma_n = tn^{-1/2}$?
(b) If in (a) we were testing the simple null hypothesis $H_s : \theta = \theta_0$ versus $K_2 : \theta \neq \theta_0$, briefly describe the limiting behavior of the three statistics corresponding to those in (a) for a fixed alternative.
(c) Consider estimation of the function

$$q(\theta) = \nu(P_\theta) = P_\theta(Y > 1|Z = 1) = \frac{P_\theta(Y > 1, Z = 1)}{P_\theta(Z = 1)}.$$

Compute $q(\theta)$ explicitly as a function of θ.
(d) Suggest a natural empirical estimator of this (conditional) probability which does not rely on the exponential model. If this estimator is called $\tilde{\nu}_n$, show that $\tilde{\nu}_n$ is asymptotically linear and find its influence function ψ explicitly.
(e) Find the efficient influence function $\tilde{l}_\nu$ for estimation of $\nu(P_\theta)$ assuming the Poisson model.
(f) Describe the relationship between ψ and $\tilde{l}_\nu$ geometrically.