Fall 2009: STAT 581.
Problem Set #6

Arseni Seregin

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Problem 2(d)

Using delta method we have:

$$\sqrt{n}\left(\begin{pmatrix}\log \mathbb{F}_n^{-1}(r)\\ \log \mathbb{F}_n^{-1}(s)\end{pmatrix} - \begin{pmatrix}\log F^{-1}(r)\\ \log F^{-1}(s)\end{pmatrix}\right) \to_d D_1\mathcal{N}(0,\Sigma),$$

where:

$$D_1 \equiv \begin{pmatrix}(\log F^{-1})'(r) & 0\\ 0 & (\log F^{-1})'(s)\end{pmatrix}$$
$$\Sigma \equiv \begin{pmatrix}r(1-r) & r(1-s)\\ r(1-s) & s(1-s)\end{pmatrix}$$

Let us denote:

$$\psi(x) = \log\log 1/(1-x).$$

One can notice that function ψ is a monotone one-to-one map from $(0,1)$ to $\mathbb{R}$. We have:

$$\begin{pmatrix}\log \mathbb{F}_n^{-1}(r)\\ \log \mathbb{F}_n^{-1}(s)\end{pmatrix} = A\begin{pmatrix}\log \hat{\alpha}_n\\ 1/\hat{\beta}_n\end{pmatrix}$$
$$\begin{pmatrix}\log F^{-1}(r)\\ \log F^{-1}(s)\end{pmatrix} = A\begin{pmatrix}\log \alpha\\ 1/\beta\end{pmatrix},$$

where:

$$A \equiv \begin{pmatrix}1 & \psi(r)\\ 1 & \psi(s)\end{pmatrix}.$$

Therefore:

$$D_1 = \frac{1}{\beta}\begin{pmatrix}\psi'(r) & 0\\ 0 & \psi'(s)\end{pmatrix}$$

where:

$$\psi'(x) = \frac{1}{(1-x)\log 1/(1-x)}$$

Using delta method again we obtain:

$$\sqrt{n}\left(\begin{pmatrix}\log \hat{\alpha}_n\\ 1/\hat{\beta}_n\end{pmatrix} - \begin{pmatrix}\log \alpha\\ 1/\beta\end{pmatrix}\right) \to_d A^{-1}D_1\mathcal{N}(0,\Sigma),$$

and:

$$\sqrt{n}\left(\begin{pmatrix}\hat{\alpha}_n\\ \hat{\beta}_n\end{pmatrix}-\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right)\to_d D_2^{-1}A^{-1}D_1\mathcal{N}(0,\Sigma)\equiv\mathcal{N}(0,\Lambda),$$

where:

$$A^{-1}=\frac{1}{\psi(s)-\psi(r)}\begin{pmatrix}\psi(s) & -\psi(r)\\ -1 & 1\end{pmatrix}$$
$$D_2\equiv\begin{pmatrix}1/\alpha & 0\\ 0 & -1/\beta^2\end{pmatrix}$$

Notice that:

$$\Sigma=-u_1u_1^T+ru_2u_2^T+(s-r)u_3u_3^T=U^TSU,$$

where:

$$u_1\equiv\begin{pmatrix}r\\ s\end{pmatrix},\ u_2\equiv\begin{pmatrix}1\\ 1\end{pmatrix},\ u_3\equiv\begin{pmatrix}0\\ 1\end{pmatrix}$$
$$U=\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}=\begin{pmatrix}r & s\\ 1 & 1\\ 0 & 1\end{pmatrix}$$
$$S=\begin{pmatrix}-1 & 0 & 0\\ 0 & r & 0\\ 0 & 0 & s-r\end{pmatrix}$$

Now we compute:

$$V\equiv UD_1\left(A^{-1}\right)^TD_2^{-1}=\frac{1}{\beta(\psi(s)-\psi(r))}\begin{pmatrix}r & s\\ 1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi'(r) & 0\\ 0 & \psi'(s)\end{pmatrix}\begin{pmatrix}\psi(s) & -1\\ -\psi(r) & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & -\beta^2\end{pmatrix}$$
$$=\frac{1}{\beta(\psi(s)-\psi(r))}\begin{pmatrix}\alpha(r\psi'(r)\psi(s)-s\psi'(s)\psi(r)) & \beta^2(r\psi'(r)-s\psi'(s))\\ \alpha(\psi'(r)\psi(s)-\psi'(s)\psi(r)) & \beta^2(\psi'(r)-\psi'(s))\\ \alpha\psi'(s)\psi(r) & \beta^2\psi'(s)\end{pmatrix}$$

Finally:

$$\Lambda=V^TSV$$
$$\Lambda_{11}=\alpha^2\frac{-(r\psi'(r)\psi(s)-s\psi'(s)\psi(r))^2+r(\psi'(r)\psi(s)-\psi'(s)\psi(r))^2+(s-r)(\psi'(s)\psi(r))^2}{\beta^2(\psi(s)-\psi(r))^2}$$
$$\Lambda_{22}=\beta^2\frac{-(r\psi'(r)-s\psi'(s))^2+r(\psi'(r)-\psi'(s))^2+(s-r)\psi'(s)^2}{(\psi(s)-\psi(r))^2}$$
$$\Lambda_{12}=-\alpha\frac{-(r\psi'(r)\psi(s)-s\psi'(s)\psi(r))(r\psi'(r)-s\psi'(s))+\dots}{(\psi(s)-\psi(r))^2}.$$

We can rewrite:

$$\Lambda_{22}=\beta^2\frac{r(1-r)\psi'(r)^2-2r(1-s)\psi'(r)\psi'(s)+s(1-s)\psi'(s)^2}{(\psi(s)-\psi(r))^2}.$$

This function is relatively smooth and we can try brute force minimization:

```
nr = 1000
r = (1:(nr-1))/nr
psi.r = log(-log(1-r))
d.psi.r = -1/((1-r)*log(1-r))

ns = 1000
s = (1:(ns-1))/ns
psi.s = log(-log(1-s))
d.psi.s = -1/((1-s)*log(1-s))

variance = 10^10
rmin = -1;
smin = -1;
for (i in (1:(nr-2)))
{
        for (j in ((i+1):(ns-1)))
        {
                newval = (1-r[i])*r[i]*d.psi.r[i]^2;
                newval = newval - 2*(1-s[j])*r[i]*d.psi.r[i]*d.psi.s[j];
                newval = newval + (1-s[j])*s[j]*d.psi.s[j]^2;
                newval = newval/(psi.s[j]-psi.r[i])^2;
                if(newval < val)
                {
                        variance = newval;
                        rmin = r[i];
                        smin = s[j];
                }
        }
}

rmin
smin
variance
```

which produces $r_{min} = 0.167$, $s_{min} = 0.974$ and variance $0.916\beta^2$.