STATISTICS 581:
Solutions, Day 1 Quiz, Fall, 2008

1. **State** Chebychev's inequality (for a random variable X with $\mu = E(X)$ and $\sigma^2 = Var(X)$).
Solution: If X is a random variable with mean μ and variance σ^2, then for every $t > 0$

$$P(|X - \mu| > t) \le \frac{E(X-\mu)^2}{t^2} = \frac{\sigma^2}{t^2}.$$

2. Use Chebychev's inequality to prove the WLLN: if $X_1, \ldots, X_n$ are i.i.d. with $EX_1 = \mu$ and $EX_1^2 < \infty$, then $\overline{X}_n \to_p \mu$.
Solution: Since $E(\overline{X}_n) = \mu$ and $Var(\overline{X}_n) = \sigma^2/n$, Chebychev's inequality yields

$$P(|\overline{X}_n - \mu| > \epsilon) \le \frac{Var(\overline{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0$$

as $n \to \infty$ for every $\epsilon > 0$. That is, $\overline{X}_n \to_p \mu$.

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. Poisson (λ) random variables.
(a) What is the distribution of $\sum_{i=1}^n X_i$?
(b) Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$. Does $\sqrt{n}(\overline{X}_n - \lambda)$ converge in distribution? If so, to what?
(c) Does $\sqrt{n}(\overline{X}_n^{1/2} - \lambda^{1/2})$ converge in distribution? If so, to what?
Solution: (a) $\sum_{i=1}^n X_i \sim \text{Poisson}(n\lambda)$.
(b) Yes. Since the X_i's are i.i.d. with $E(X_i) = \lambda$ and $Var(X_i)$ we have

$$\sqrt{n}(\overline{X}_n - \lambda) \to Z \sim N(0, \lambda)$$

by the ordinary CLT.
(c) Yes. Since $g(x) = \sqrt{x}$ has $g'(x) = (1/2)x^{-1/2}$, the delta-method yields

$$\begin{aligned} \sqrt{n}(\overline{X}_n^{1/2} - \lambda^{1/2}) &= \sqrt{n}(g(\overline{X}_n) - g(\lambda)) \to_d g'(\lambda)Z \\ &= \frac{1}{2\sqrt{\lambda}} Z \sim N(0, 1/4). \end{aligned}$$

4. Suppose that (X, Y) have joint distribution function on $[0,1]^2$ given by

$$F(x, y) = xy\{1 + \theta(1-x)(1-y)\}$$

for $0 \le x, y \le 1$ and $|\theta| \le 1$.
(a) What is the density function f of (X, Y)?
(b) What is the (marginal) distribution function of X? [Does it have a name?]
(c) Compute EX, EY, $Var(X)$ and $Var(Y)$.
(d) Compute $Cov(X, Y)$ and $Corr(X, Y)$.

Solution: (a) The density function f is given by

$$\begin{aligned} f(x,y) &= \frac{\partial^2 F}{\partial x \partial y}(x,y) \\ &= \frac{\partial}{\partial x}\{x(1+\theta(1-x)(1-y)+xy(-\theta(1-x))\} \\ &= \frac{\partial}{\partial x}\{x(1+\theta(1-x)(1-2y))\} \\ &= (1+\theta(1-x)(1-2y)) - \theta x(1-2y) \\ &= 1+\theta(1-2x)(1-2y). \end{aligned}$$

(b) Taking $y=1$ in the joint distribution function gives $F(x,1) = x \cdot 1\{1+\theta(1-x)\cdot 0\} = x$ for $0 \le x \le 1$. Thus the marginal distribution of X is Uniform$(0,1)$. By symmetry the marginal distribution of Y is also Uniform$(0,1)$.
(c) $E(X) = 1/2 = E(Y)$, $Var(X) = 1/12 = Var(Y)$.
(d) Now

$$\begin{aligned} E(XY) &= \int_0^1 \int_0^1 xyf(x,y)dxdy \\ &= \int_0^1 \int_0^1 xy\{1+\theta(1-2x)(1-2y)\}dxdy \\ &= \left(\int_0^1 xdx\right)^2 + \theta\left(\int_0^1 x(1-2x)dx\right)^2 \\ &= (1/2)^2 + \theta(-1/6)^2 = 1/4 + \theta(1/36). \end{aligned}$$

Thus $Cov(X,Y) = E(XY) - E(X)E(Y) = \theta/36$ and

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\theta/36}{1/12} = \frac{\theta}{3}.$$

Notes: This is the Morgenstern "copula" model. This particular model is not terribly useful since it does not allow a full range of the correlation between X and Y: note that the correlation ranges between $-1/3$ and $1/3$ as θ ranges from -1 to $+1$.

5. Suppose that U is a random variable with a Uniform$(0,1)$ distribution. For each integer $n \ge 1$ define $X_n = n^{3/5} 1_{[0,1/n]}(U)$.
(a) Does $X_n \to_d X$? (If so, identify the distribution of the limiting variable X.)
(b) Does $X_n \to_p X$? (If so, identify the limit variable X.)
(c) Does $X_n \to_1 X$ (i.e. in L_1, or "first-mean")?
(d) Does $X_n \to_2 X$ (i.e. in L_2, or "mean square")?
Solution: (a) & (b) Yes to both: $P(X_n > \epsilon) \le P(U \le 1/n) = n^{-1} \to 0$ for every $\epsilon > 0$, so $X_n \to_p 0 \equiv X$ with X the random variable which is 0 with probability 1. Since $\to_p$ implies $\to_d$, $X_n \to_d 0$ also.
(c) Yes, $X_n \to_1 0$ since

$$E|X_n - X| = EX_n = n^{3/5} E1_{[0,1/n]}(U) = n^{3/5} n^{-1} = n^{-2/5} \to 0.$$

(d) No, since

$$E|X_n - X|^2 = EX_n^2 = n^{6/5} E1_{[0,1/n]}(U) = n^{6/5} n^{-1} = n^{1/5} \to \infty.$$

6. Suppose that X, $X_1, \ldots, X_n, \ldots$ are independent and identically distributed Exponential(θ) random variables (i.e. $P(X > x) = \exp(-\theta x)$ for $x \geq 0$).
There was a typo in the quiz here. My apologies. "Rayleigh" should have been replaced by "Exponential" as above.
Let $T_n = X_1 + \ldots + X_n$.
(a) What is the distribution of X^2?
(b) What is the distribution of T_n?
(c) Compute $E(X)$ and $Var(X)$.
(d) Does $\overline{X}_n = n^{-1}T_n \rightarrow_p$ something? If so, what is "something"?
(e) Does $\sqrt{n}(\overline{X}_n - \theta^{-1}) \rightarrow_d$ something? If so, what is "something"?
(f) What is the Cramér-Rao bound for unbiased estimators of θ?

Solution: (a) $P(X^2 > y) = P(X > \sqrt{y}) = \exp(-\theta y^{1/2})$, so $Y = X^2$ has distribution function given by $F_{X^2}(y) = 1 - \exp(-\theta y^{1/2})$.
(b) $T_n = X_1 + \cdots + X_n \sim \text{Gamma}(n, \theta)$.
(c) Now

$$E(X) = \int_0^\infty P(X > x)dx = \int_0^\infty \exp(-\theta x)dx = 1/\theta,$$

and

$$E(X^2) = \int_0^\infty 2xP(X > x)dx = 2\int_0^\infty x\exp(-\theta x)dx = 2/\theta^2.$$

Thus $Var(X) = E(X^2) - (E(X))^2 = 2/\theta^2 - 1/\theta^2 = 1/\theta^2$.
(d) Yes: $n^{-1}T_n \rightarrow_p E(X_1) = 1/\theta$ by the WLLN.
(e) Yes:

$$\sqrt{n}(\overline{X}_n - \theta^{-1}) \rightarrow_d N(0, 1/\theta^2)$$

by the ordinary CLT.
(f) The density of X is $p_\theta(x) = \theta\exp(-\theta x)1_{(0,\infty)}(x)$, so $\log p_\theta(x) = \log\theta - \theta x$, and the score function for θ is

$$\dot{l}_\theta(x) = \frac{\partial}{\partial\theta}\log p_\theta(x) = \frac{1}{\theta} - x.$$

Thus the information for θ is given by

$$I(\theta) = E_\theta \dot{l}_\theta^2(X) = E_\theta(\theta^{-1} - X)^2 = Var_\theta(X) = \theta^{-2},$$

and the Cramér - Rao lower bound for unbiased estimators of θ is given by $1/(nI(\theta)) = 1/(n\theta^{-2}) = \theta^2/n$.