Statistics 581, Problem Set 7

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Reading: Chapter 3, Section 2;
Ferguson, ACILST, Chapter 19, pages 126-132, Chapter 20, pages 133-134;
Lehmann and Casella, pages 113-129, and 439- 443.
Due: Wednesday, November 12, 2008.

1. Compute and plot the *score for location*, $-(f'/f)(x)$ when:
   A. $f(x) = \phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, (normal or Gaussian);
   B. $f(x) = \exp(-x)/(1+\exp(-x))^2$, (logistic);
   C. $f(x) = \frac{1}{2}\exp(-|x|)$, (double exponential);
   D. $f = t_k$, the $t-$distribution with k degrees of freedom;
   E. $f(x) = \exp(-x)\exp(-\exp(-x))$, Gumbel or extreme value.

2. Compute $I_f = \int (f'(x)/f(x))^2 f(x)dx$, the information for location, for each of the densities in problem 1.

3. Consider the two parameter location-scale model

$$\mathcal{P} = \{P_\theta : \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta\}$$

   where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right),$$

   and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ.
   (a) Calculate the information matrix $I(\theta)$ for θ.
   (b) For which of the densities in A-E of problem 1 is $I_{12}(\theta)$ not zero?

4. Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0,\infty) \times (0,\infty) \equiv \Theta .$$

   Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0; here $F_\theta(x) \equiv P_\theta(X \le x)$.
   (i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page 127, Table 6.1
   (ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
   (iii) Find the efficient influence functions for estimation of q_A and q_B.
   (iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators $\overline{X}_n$ and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.

5. Lehmann and Casella, TPE, Problem 6.6, page 142.

6. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \le m < k$. Let $\dot{l} = \dot{l}_\theta = (\dot{l}_1, \dot{l}_2)$ be the corresponding partition of the (vector of) scores $\dot{l}$, and, with $\widetilde{l} \equiv I^{-1}(\theta)\dot{l}$, the *efficient influence function* for θ, let $\widetilde{l} = (\widetilde{l}_1, \widetilde{l}_2)$ be the corresponding partition of $\widetilde{l}$. In both cases, $\dot{l}_1$, $\widetilde{l}_1$ are $m-$vectors of functions, and $\dot{l}_2$, $\widetilde{l}_2$ are $k-m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where I_{11} is $m \times m$, I_{12} is $m \times (k-m)$, I_{21} is $(k-m) \times m$, I_{22} is $(k-m) \times (k-m)$. Also write

$$I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}.$$

Verify that:
A. $I^{11} = I_{11\cdot 2}^{-1}$ where $I_{11\cdot 2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21}$,
$I^{22} = I_{22\cdot 1}^{-1}$ where $I_{22\cdot 1} \equiv I_{22} - I_{21}I_{11}^{-1}I_{12}$,
$I^{12} = -I_{11\cdot 2}^{-1}I_{12}I_{22}^{-1}$,
$I^{21} = -I_{22\cdot 1}^{-1}I_{21}I_{11}^{-1}$.
This amounts to formulas (3) and (4) of section 3.2, page 14.
B. Verify that
$\widetilde{l}_1 = I^{11}\dot{l}_1 + I^{12}\dot{l}_2 = I_{11\cdot 2}^{-1}(\dot{l}_1 - I_{12}I_{22}^{-1}\dot{l}_2)$, and
$\widetilde{l}_2 = I^{21}\dot{l}_1 + I^{22}\dot{l}_2 = I_{22\cdot 1}^{-1}(\dot{l}_2 - I_{21}I_{11}^{-1}\dot{l}_1)$.

7. **Optional bonus problem:** [This is example 7.2.5 and 7.2.7 in Lehmann and Casella, TPE, section 6.2; also see problems 6.2.12 - 6.2.14, Lehmann and Casella, TPE, page 501.] Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, 1)$ so $I(\theta) = 1$. Let $0 < a < 1$ and define $T_n \equiv \overline{X}_n 1_{[|\overline{X}_n| \ge n^{-1/4}]} + a\overline{X}_n 1_{[|\overline{X}_n| < n^{-1/4}]}$. This is Hodges superefficient estimator of θ.
(a) Show that $\sqrt{n}(T_n - \theta) \to_d N(0, V(\theta))$ where

$$V(\theta) = \begin{cases} 1, & \theta \neq 0 \\ a^2, & \theta = 0 \end{cases}$$

(b) Show that T_n is *not a regular estimator* of θ at $\theta = 0$, but that it is regular at every $\theta \neq 0$. If $\theta_n = t/\sqrt{n}$, find the limiting distribution of $\sqrt{n}(T_n - \theta_n)$ under P_{θ_n}.
C. For $\theta_n = t/\sqrt{n}$ show that

$$R_n(\theta_n) = nE_{\theta_n}(T_n - \theta_n)^2 \to E(aZ + t(a-1))^2 = a^2 + t^2(1-a)^2$$

where $Z \sim N(0,1)$. This is *larger* than 1 if $t^2 > (1+a)/(1-a)$, and hence superefficiency also entails *worse* risks in a local neighborhood of the point(s) where the asymptotic variance is smaller.