

Statistics 581, Problem Set 2

Wellner; 10/1/2008

Reading: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 1-25.

Due: Wednesday, October 8, 2008.

1. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$\text{Var}(Y) = E\{\text{Var}(Y|X)\} + \text{Var}\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}.$$

- (b) Interpret (a) geometrically.
- (c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $\text{Var}(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

2. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.

Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

3. Ferguson, ACILST, #1, page 11.

Let X_1, X_2, \dots be i.i.d. random variables with densities $f(x) = \alpha x^{-(\alpha+1)} 1_{(1,\infty)}(x)$.

- (a) For what values of $\alpha > 0$ and $r > 0$ is it true that $n^{-1}X_n \rightarrow_r 0$?
- (b) For what values of $\alpha > 0$ is it true that $n^{-1}X_n \rightarrow_{a.s.} 0$?
- (c) If X_1, X_2, \dots are independent with X_n having density $f_n(x) = \alpha_n x^{-(\alpha_n+1)} 1_{(1,\infty)}(x)$ for $n = 1, 2, \dots$, Find the limit of $n^{-2}EX_n^2$ when $\alpha_n = 2 + n^{-\gamma}$ for $\gamma \in \mathbb{R}$.

4. (a) Ferguson, ACILST, #4, page 6: Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow 1$.
(b) Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow \infty$.
(c) Give an example of a sequence of random variables X_n for which $X_n \rightarrow_p 0$ but $X_n \rightarrow_{a.s.} 0$ fails.

5. Continuation of problem 3, Problem set 1: Suppose that $X \sim F$ on $R^+ \equiv [0, \infty)$, $Y \sim G$ on R^+ , and X and Y are independent random variables, but that only Y and $\Delta = 1\{X \leq Y\}$ are observed. (This is called “type 1 interval censored data”, or “current status data”.)

(a) Find the joint distribution of (Y, Δ) by first computing the distribution of Δ conditional on Y , and then computing the two sub-distribution functions $F_1(y) \equiv P(Y \leq y, \Delta = 1)$ and $F_0(t) \equiv P(Y \leq y, \Delta = 0)$.

(b) If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, are Y and Δ independent?

6. (a) Lehmann and Casella, #3.5, page 64.

(b) Lehmann and Casella, #3.6, page 64.

(c) Lehmann and Casella, #3.7, page 64.

7. **Optional bonus problem:**

(a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W , distribution function F_W , and inverse distribution function F_W^{-1} explicitly.

(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(0, 2\pi)$.

(c) Use the results of (a) and (b) to show (using Theorem 2.3.4) how to use two independent $\text{Uniform}(0, 1)$ random variables U and V to generate two standard normal random variables.