

Statistics 581, Problem Set 10

Wellner; 11/26/2008

Reading: Chapter 4, Sections 1-4;

Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;

Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.

Due: Wednesday, December 3, 2008.

1. Suppose that (as in Lemma 5.2, page 38, Chapter 3 Notes) P and Q are two probability measures on a measurable space $(\mathcal{X}, \mathcal{A})$ with densities p and q with respect to a σ -finite dominating measure μ , and P^n and Q^n denote the corresponding product measures on $(\mathcal{X}^n, \mathcal{A}_n)$ (of X_1, \dots, X_n i.i.d. as P or Q respectively).
 - (a) What is the relationship between $K(P^n, Q^n)$ and $K(P, Q)$, if any?
 - (b) If P is the Normal($0, \sigma^2$) distribution and Q is the Normal(μ, σ^2) distribution, compute $K(P, Q)$, $\rho(P, Q) = \int \sqrt{pq} d\mu$, and $H^2(P, Q)$.
 - (c) Use the results of (a) and (b) together with Lemma 5.2 to calculate $K(P^n, Q^n)$, $\rho(P^n, Q^n)$, and $H^2(P^n, Q^n)$ when P and Q are as in (b).
 - (d) Find a sequence μ_n so that, with Q_n being the Normal distribution with mean μ_n , the quantities $K(P^n, Q_n^n)$, $\rho(P^n, Q_n^n)$, and $H^2(P^n, Q_n^n)$ converge to finite limits as $n \rightarrow \infty$.
2. Consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities

$$p_\theta(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{x}{\alpha}\right)^\beta\right) 1_{[0, \infty)}(x)$$

where $\theta \equiv (\alpha, \beta) \in (0, \infty) \times (0, \infty) \subset R^2$. Suppose that X, X_1, \dots, X_n are i.i.d. with density function p_θ .

- (a) If $X \sim P_\theta \in \mathcal{P}$, show that the distributions of $\log X$ form a location and scale family from a Gumbel (extreme value) density on R .
- (b) Use the result of A to construct method of moments estimators or quantile based estimators $\bar{\theta}_n$ of $\theta = (\alpha, \beta)$.
- (c) Show that the method of moments or quantile estimators $\bar{\theta}_n$ of θ are asymptotically normal, and find the asymptotic distribution; i.e. show that

$$\sqrt{n}(\bar{\theta}_n - \theta) \rightarrow_d N_2(0, \Sigma) \quad \text{for some } \Sigma.$$

[We will use these estimators as “starting points” approximate (or one-step) maximum likelihood estimators in the next problem.]

3. (Problem #2, continued).
 - (a) Does a maximum likelihood estimate of $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ exist? Is it unique? (See Lehmann and Casella, Example 6.1, page 468.)
 - (b) Compute an approximate (one - step) maximum likelihood estimate $\check{\theta}$ of θ using

the method of moment (or quantile) estimators $\bar{\theta}_n$ as the preliminary estimators based on the following data (with $n = 12$):

1, 1, 2, 3, 12, 25, 46, 56, 79, 125, 323, 417.

[These are failure times in seconds for “breakdown” of an insulating fluid between two electrodes subject to a voltage of 40 kV. – from Nelson, *Applied Life Data Analysis*, page 252, but with some modifications or “recording errors”.]

(c) Compute the maximum likelihood estimator $\hat{\theta}_n$, and compare it with the one step estimator computed in E.

4. (a) Ferguson, ACLST, page 139, problem 3.
(b) What if Ferguson’s density $f(x|\theta)$ with $\theta \in (0, 1)$ is replaced by $\theta = (\gamma, \eta) \in (0, 1) \times (0, \infty)$ and

$$f(x|\theta) \equiv f(x|\gamma, \eta) = \{(1 - \gamma)e^{-x} + \gamma\eta^2 x \exp(-\eta x)\}1_{[0, \infty)}(x)?$$

Can you estimate γ and η by the method of moments? Can you improve method of moment estimators via one-step estimators?

5. Ferguson, ACLST, page 118, problem 3. [Also see Lehmann and Casella, Example 7.9, page 482.]
6. **Optional bonus problem:** Ferguson, ACLST, page 149, problem 2 modified as follows:
(a) Find the LR test statistic of the null hypothesis $H_0 : \mu = c\theta$ for any fixed number $c > 0$, and find the asymptotic distribution of the LR statistic under H_0 .
(b) Does the theory of our chapter 4 (or Ferguson’s chapter 22) apply directly?
(c) Does the local asymptotic power of your test depend on c ?