STATISTICS 581:
Solutions, Day 1 Quiz, Fall 2006

1. **State** the (classical) Central Limit Theorem.

Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$, then

$$\sqrt{n}(\overline{X}_n - \mu) \to_d N(0, \sigma^2)\,.$$

In other words, if $Z \sim N(0,1)$ and Φ denotes the distribution function of a standard normal random variable,

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) dy\,,$$

then, for all $t \in R$,

$$P(\sqrt{n}(\overline{X}_n - \mu) \le t) \to P(\sigma Z \le t) = \Phi(t/\sigma)\,. \quad (1)$$

[Note: the hypothesis $\sigma^2 < \infty$ is also necessary: if i.i.d. random variables X_i satisfy (1) then $\sigma^2 < \infty$.]

2. **State** the weak law of large numbers.

Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E|X_1| < \infty$ and $E(X_1) = \mu$, then $\overline{X}_n = n^{-1}\sum_1^n X_i \to_p \mu$.

3. Suppose that X has density $f(x) = 3x^{-4} 1_{[1,\infty)}(x)$.
(a) For what values of $r \in R$ is it true that $E(X^r) < \infty$?
(b) For the values of r for which the expectation is finite, compute it explicitly.
(c) If $X_1, X_2, \ldots$ are i.i.d. with the same density as X, does the law of large numbers hold?
(d) If $X_1, X_2, \ldots$ are i.i.d. with the same density as X, does the central limit theorem hold?

Solution:
(a) $E(X^r) = \int_1^\infty x^r 3x^{-4} dx = \int_1^\infty 3x^{r-4} dx = 3x^{r-3}/(r-3)|_1^\infty = 3/(3-r) < \infty$ if $r < 3$. If $r > 3$ then $E(X^r) = \infty$ by the same calculation, while if $r = 3$, then $E(X) = \int_1^\infty 3x^{-1} dx = \log x|_1^\infty = \infty$.
(b) As shown in (a) $E(X^r) = 3/(3-r)$ for $r < 3$.
(c) Yes, since $E(X) = 3/2$.
(d) Yes, since $E(X^2) = 3/1 = 3$.

4. **Define** what is meant by:
A. X_n converges in distribution to X for random variables X and X_n, $n \ge 1$.
B. X_n converges in probability to X.

Solution: A. X_n converges in distribution to X if

$$F_n(x) = P(X_n \le x) \to P(X \le x) = F(x)$$

for all $x \in C_F = \{x \in R: \ F$ is continuous at $x\}$.
B. X_n converges in probability to X if

$$P(|X_n - X| > \epsilon) \to 0 \qquad \text{as } n \to \infty \text{ for every } \epsilon > 0\,.$$

5. Suppose that U is a random variable with a Uniform$(0,1)$ distribution. For each integer $n \geq 1$ define $X_n = \sqrt{n} 1_{[0,1/n]}(U)$.
(a) Does $X_n \to_d X$? (If so, identify the distribution of the limiting variable X.)
(b) Does $X_n \to_p X$? (If so, identify the limit variable X.)
(c) Does $X_n \to_1 X$ (i.e. in L_1, or "first mean")?
(d) Does $X_n \to_2 X$ (i.e. in L_2, or "mean square"?

Solution: Here it is easier to answer (b) first, then (a):
(b) Note that for any $\epsilon \in (0,1)$

$$P(|X_n| > \epsilon) = P(U \leq 1/n) = 1/n \to 0\,.$$

Hence $X_n \to_p 0$.
(a) Since convergence in probability implies convergence in distribution, the result of (b) implies that $X_n \to_d 0$. [In fact, $X_n \to_{a.s.} 0$: for $U(\omega) \in (0,1]$ we have $1/n < U(\omega)$ for all $n \geq N(\omega)$ sufficiently large, and hence $X_n(\omega) = 0$ for $n \geq N(\omega)$. But $P(U \in (0,1]) = 1$, so it follows that $X_n \to_{a.s.} 0$. Since $\to_{a.s.}$ implies $\to_p$, this also yields the desired conclusion(s).]
(c) The expectation is

$$E(X_n) = E\{\sqrt{n} 1_{[0,1/n]}(U)\} = \sqrt{n} P(0 \leq U \leq 1/n) = \sqrt{n}(1/n) = n^{-1/2} \to 0.$$

Thus $X_n \to_1 0$. (d) The expectation of the square is

$$E(X_n^2) = E\{n 1_{[0,1/n]}(U)\} = nP(0 \leq U \leq 1/n) = n(1/n) = 1 \nrightarrow 0,$$

so X_n does not converge to 0 in L_2 (or mean-square).

6. Suppose that $X,\ X_1, \ldots, X_n, ...$ are independent and identically distributed Rayleigh(θ) random variables (i.e. $P(X > x) = \exp(-\theta x^2)$ for $x \geq 0$). Let $T_n = X_1^2 + \ldots + X_n^2$.
(a) What is the distribution of X^2? What is the distribution of T_n?
(b) Compute $E(X^2)$ and $Var(X^2)$.
(c) Does $\overline{X^2}_n = n^{-1} T_n \to_p$ something? If so, what is "something"?
(c) Does $\sqrt{n}(\overline{X^2}_n - \theta^{-1}) \to_d$ something? If so, what is "something"?
(d) What is the Cramér-Rao bound for unbiased estimators of θ?

Solution:
(a) Now

$$P(X^2 > y) = P(X > \sqrt{y}) = \exp(-\theta(\sqrt{y})^2) = \exp(-\theta y)$$

for $y \geq 0$; i.e. X has an Exponential distribution with (rate) parameter θ. Hence $X_1^2, X_2^2, \ldots, X_n^2$ are i.i.d. Exponential (θ) random variables, and $T_n = X_1^2 + \cdots + X_n^2 \equiv$

$Y_1 + \cdots + Y_n$ has a Gamma(n, θ) distribution.
(b) For arbitrary $r > 0$,

$$\begin{aligned} E(X^r) &= \int_0^\infty y^{r/2}\theta \exp(-\theta x) dx \\ &= \theta^{-r/2} \int_0^\infty (\theta y)^{r/2} \exp(-\theta y) d(\theta y) \\ &= \theta^{-r/2} \int_0^\infty z^{r/2} \exp(-z) dz \qquad \text{by the change of variables } z = \theta y \\ &= \theta^{-r/2}\Gamma(r/2+1). \end{aligned}$$

Therefore $E(X^2) = \theta^{-1}\Gamma(2) = \theta^{-1}1! = \theta^{-1}$ while $E(X^4) = \theta^{-2}\Gamma(3) = \theta^{-2}2$. It follows easily that $Var(X^2) = Var(Y) = \theta^{-2}$.
(c) By the weak law of large numbers, $\overline{X^2}_n \to_p \theta^{-1}$.
(d) By the central limit theorem, $\sqrt{n}(\overline{X}_n - \theta^{-1}) \to_d N(0, 1/\theta^2)$.
(e) The Cramér - Rao bound for unbiased estimators of θ is given by $1/(nI(\theta))$ where

$$I(\theta) = E\{\dot{\mathbf{l}}_\theta^2(X^2)\} = E\left\{\frac{1}{\theta} - Y\right\}^2 = Var(Y) = \theta^{-2}.$$

Hence the Cramér - Rao lower bound in this case is θ^2/n. Alternatively,

$$I(\theta) = -E\{\ddot{\mathbf{l}}_{\theta\theta}(Y)\} = -E\{-1/\theta^2\} = 1/\theta^2.$$