

Statistics 581, Problem Set 4 Solutions

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1. Suppose that $\underline{N}_n \sim \text{Mult}_k(n, \underline{p})$ and $\hat{\underline{p}} = \underline{N}_n/n$. Suppose that the true \underline{p} is $\underline{p}_n = \underline{p}_0 + n^{-1/2}\underline{c}$ where $\underline{1}^T \underline{c} = 0$. Use the Cramér - Wold device together with either the Liapunov or the Lindeberg-Feller CLT to show that

$$\underline{Z}_n = \left(\frac{N_{n,1} - np_{n,1}}{\sqrt{np_{0,1}}}, \dots, \frac{N_{n,k} - np_{n,k}}{\sqrt{np_{0,k}}} \right)$$

satisfies $\underline{Z}_n \rightarrow_d \underline{Z}$ where $\underline{Z} \sim N_k(0, I - \sqrt{p_0}\sqrt{p_0}^T)$. (It therefore follows, as outlined in class, that the chi-square statistic $Q_n \rightarrow_d \chi_{k-1}^2(\delta)$ with $\delta = \sum_{j=1}^k c_j^2/p_{0,j}$ under the local alternative \underline{p}_n .)

Solution: We argued heuristically in class that when the true $\underline{p} = \underline{p}_n = \underline{p}_0 + \underline{c}n^{-1/2}$, then

$$(1) \quad \underline{Z}_n \equiv \text{diag}(1/\sqrt{\underline{p}_0})n^{1/2}(\hat{\underline{p}} - \underline{p}_n) \rightarrow \underline{Z} + \underline{d} \sim N_k(\underline{d}, \Sigma)$$

where $\underline{d} = \text{diag}(1/\sqrt{\underline{p}_0})\underline{c}$ and $\Sigma = I - \sqrt{\underline{p}_0}\sqrt{\underline{p}_0}^T$. To prove that (1) holds, we can use the Cramér-Wold device and the Liapunov CLT. Fix $\underline{a} \in R^k$. Then we want to show that

$$\underline{a}^T \sqrt{n}(\hat{\underline{p}} - \underline{p}_n) \rightarrow_d N(0, \underline{a}^T(\text{diag}(\underline{p}_0) - \underline{p}_0 \underline{p}_0^T)\underline{a}).$$

But since $\underline{N}_n = \sum_{i=1}^n \underline{V}_{ni}$ where $\underline{V}_{ni} \sim \text{Mult}_k(1, \underline{p}_n)$ are i.i.d. for each n , we can write

$$\begin{aligned} \underline{a}^T \sqrt{n}(\hat{\underline{p}} - \underline{p}_n) &= \sum_{i=1}^n \sum_{j=1}^k a_j (V_{ni,j} - p_{nj}) / \sqrt{n} \\ &\equiv \sum_{i=1}^n X_{ni} \end{aligned}$$

where the X_{ni} 's have $\mu_{ni} = E(X_{ni}) = 0$,

$$\sigma_{ni}^2 = \text{Var}(X_{ni}) = \underline{a}^T(\text{diag}(\underline{p}_n) - \underline{p}_n \underline{p}_n^T)\underline{a}/n$$

and

$$\gamma_{ni} = E|X_{ni}|^3 = n^{-3/2} \sum_{j'=1}^k \left\{ \left| a_{j'}(1 - p_{nj'}) + \sum_{j \neq j', j=1}^k a_j(0 - p_{nj}) \right|^3 \right\} p_{nj'}$$

so that

$$\sigma_n^2 = \sum_1^n \sigma_{ni}^2 = \underline{a}^T(\text{diag}(\underline{p}_n) - \underline{p}_n \underline{p}_n^T)\underline{a} \rightarrow \underline{a}^T \Sigma \underline{a}$$

while

$$\begin{aligned}
\gamma_n &= \sum_1^n \gamma_{ni} \\
&= n^{-1/2} \sum_{j'=1}^k \left\{ \left| \sum_{j=1}^k a_j(1 - p_{nj}) + \sum_{j=1, j \neq j'}^k a_j(0 - p_{nj}) \right|^3 \right\} p_{nj'} \\
&\rightarrow 0 \cdot M(\underline{a}, \underline{p}_0) = 0
\end{aligned}$$

where

$$M(\underline{a}, \underline{p}_0) = \sum_{j'=1}^k \left\{ \left| \sum_{j=1}^k a_j(1 - p_{0j}) + \sum_{j=1}^k a_j(0 - p_{0j}) \right|^3 \right\} p_{0j'}.$$

hence it follows that $\gamma_n/\sigma_n^{3/2} \rightarrow 0$, and

$$\frac{\underline{a}^T \sqrt{n}(\widehat{\underline{p}}_n - \underline{p}_n)}{\sigma_n} = \frac{\sum_{i=1}^n X_{ni}}{\sigma_n} \rightarrow_d N(0, 1).$$

This implies

$$\underline{a}^T \sqrt{n}(\widehat{\underline{p}}_n - \underline{p}_n) \rightarrow_d N(0, \underline{a}^T \Sigma \underline{a}),$$

and by Cramér - Wold, this implies

$$\sqrt{n}(\widehat{\underline{p}}_n - \underline{p}_n) \rightarrow_d N_k(0, \Sigma).$$

2. Ferguson, ACILST, problem 5, page 50: (The Poisson dispersion test). A standard test of the hypothesis H_0 that a distribution is $\text{Poisson}(\lambda)$ for some λ is to reject H_0 if the ratio of the sample variance to the sample mean, S_n^2/\bar{X}_n , is too large. This test is good against alternatives whose variance is greater than the mean, such as the negative binomial distribution or any other mixture of Poisson distributions.

(a) Find the asymptotic distribution of S_n^2/\bar{X}_n for general distributions.

(b) Find the asymptotic distribution of S_n^2/\bar{X}_n under H_0 and show that it is independent of λ .

Solution: (a) We can use the result of part (a) of problem 3 of Problem Set #3. We just need to proceed as in (b) of problem 3, Problem Set #3 with $g(u, v) = v/u$. Thus we find that $\nabla g(u, v) = (-v/u^2, 1/u) = (-v/u, 1)/u$. Hence $\nabla g(\mu, \sigma^2) = (-\sigma^2/\mu, 1)/\mu$, and the limiting variance is

$$\begin{aligned}
\nabla g^T \Sigma \nabla g &= \frac{\sigma^4}{\mu^2} \left(\frac{\sigma^2}{\mu^2} - 2 \frac{\mu_3}{\mu \sigma^2} + \frac{\mu_4}{\sigma^4} - 1 \right) \\
(2) \quad &= \frac{\sigma^4}{\mu^2} \left(\frac{\sigma^2}{\mu^2} - 2 \frac{\sigma \gamma_1}{\mu} + 2 + \gamma_2 \right) \equiv V^2.
\end{aligned}$$

Thus it follows that

$$\sqrt{n} \left(\frac{S_n^2}{\bar{X}_n} - \frac{\sigma^2}{\mu} \right) \rightarrow_d N(0, V^2)$$

where V^2 is given in (2).

(b) When $X \sim \text{Poisson}(\lambda)$, $E(X) = \lambda$, $\text{Var}(X) = \lambda$, $\gamma_1 = 1/\sqrt{\lambda}$, and $\gamma_2 = 1/\lambda$. Thus we find that the asymptotic variance above is

$$\frac{\lambda^2}{\lambda^2} \left\{ \frac{\lambda}{\lambda^2} - 2 \frac{\lambda^{1/2} \lambda^{-1/2}}{\lambda} + 2 + \frac{1}{\lambda} \right\} = 2.$$

Thus it follows that under $X \sim \text{Poisson}(\lambda)$ we have

$$\sqrt{n} \left(\frac{S_n^2}{\bar{X}_n} - \frac{\sigma^2}{\mu} \right) = \sqrt{n} \left(\frac{S_n^2}{\bar{X}_n} - 1 \right) \rightarrow_d N(0, 2).$$

3. (Continuation of problem 2 above.) Suppose that $(X|\Lambda) \sim \text{Poisson}(\Lambda)$ where $\Lambda \sim \Gamma(r, b)$ with density $b^r \lambda^{r-1} \exp(-b\lambda)/\Gamma(r)$ for some $r > 0$ and $b > 0$.

(a) Show that the marginal distribution of X is Negative Binomial $(b/(1+b), r)$ with density (probability mass function)

$$P_{r,b}(X = x) = \frac{\Gamma(r+x)}{x! \Gamma(r)} \left(\frac{b}{1+b} \right)^r \frac{1}{(1+b)^x}$$

for $x = 0, 1, \dots$

(b) Show that $E(X) = r/b$ and $\text{Var}(X) = (r/b) + r/b^2 > (r/b) = E(X)$, and hence if $b \equiv b_n = \sqrt{n}/\lambda_0$ and $r \equiv r_n = \sqrt{n}$, we have, letting E_n and Var_n denote expectation and variance under (r_n, b_n) , $E_n(X) \rightarrow \lambda_0$ and $\text{Var}_n(X) \rightarrow \lambda_0$, while $\sqrt{n}(\text{Var}_n(X) - \lambda_0) \rightarrow \lambda_0^2$. (Hint: Use our results for computing the mean and variance conditionally on another random variable.)

(c) Show that if $X_n \sim \text{Negative Binomial}(b_n/(1+b_n), r_n)$ with b_n and r_n as in (b), then $X_n \rightarrow_d X_0 \sim \text{Poisson}(\lambda_0)$.

(d) Now suppose that $X_{ni} \sim \text{Negative Binomial}(b_n/(1+b_n), r_n)$ for $i = 1, \dots, n$ are independent with b_n and r_n as in (b). Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_{ni}$ and $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_{ni} - \bar{X}_n)^2$ as in problem 2. Use the results of (b) to show that under this family of local alternatives to the Poisson distribution we have

$$\sqrt{n}(S_n^2/\bar{X}_n - 1) \rightarrow_d N(c, 2)$$

for some $c \neq 0$ and find c . Use this to approximate the Power of the test in problem 2 for this particular sequence of alternatives.

Solution: (a) By direct calculation

$$\begin{aligned} P_{r,b}(X = x) &= EP(X = x|\Lambda) \\ &= \int_0^\infty \frac{\lambda^x}{x!} e^{-\lambda} \frac{b(b\lambda)^{r-1}}{\Gamma(r)} \exp(-b\lambda) d\lambda \\ &= \frac{b^r}{x! \Gamma(r)} \int_0^\infty \lambda^{x+r-1} \exp(-(1+b)\lambda) d\lambda \\ &= \frac{b^r}{x! \Gamma(r)} \frac{1}{(1+b)^{r+x}} \int_0^\infty ((1+b)\lambda)^{r+x-1} e^{-(1+b)\lambda} (1+b) d\lambda \\ &= \frac{b^r}{x! \Gamma(r)} \frac{1}{(1+b)^{r+x}} \int_0^\infty v^{r+x-1} e^{-v} dv \\ &= \frac{\Gamma(r+x)}{x! \Gamma(r)} \left(\frac{b}{1+b} \right)^r \frac{1}{(1+b)^x} \end{aligned}$$

for $x = 0, 1, 2, \dots$. That is, $X \sim \text{Negative Binomial}(p = b/(1+b), r)$.

(b) By computing conditionally on Λ we get

$$\begin{aligned} E(X) &= E(E(X|\Lambda)) = E\Lambda = r/b, \\ \text{Var}(X) &= E(\text{Var}(X|\Lambda)) + \text{Var}(E(X|\Lambda)) \\ &= E(\Lambda) + \text{Var}(\Lambda) = r/b + r/b^2. \end{aligned}$$

Thus when $b \equiv b_n = \sqrt{n}/\lambda_0$ and $r \equiv r_n = \sqrt{n}$, we have,

$$\begin{aligned} E_n(X) &= r_n/b_n = \sqrt{n}/(\sqrt{n}/\lambda_0) = \lambda_0, \quad \text{and} \\ \text{Var}_n(X) &= r_n/b_n + r_n/b_n^2 = \lambda_0 + \sqrt{n}/(n/\lambda_0^2) \\ &= \lambda_0 + \lambda_0^2/\sqrt{n} \rightarrow \lambda_0, \end{aligned}$$

while

$$\sqrt{n}(\text{Var}_n(X) - \lambda_0) = \lambda_0^2.$$

(c) Since $E_n(\Lambda) = r_n/b_n = \lambda_0$ and

$$\text{Var}_n(\Lambda) = r_n/b_n^2 = \lambda_0^2/\sqrt{n} \rightarrow 0,$$

it follows that $\Lambda_n \rightarrow_p \lambda_0$; that is the sequence of mixing distributions converges to the distribution degenerate at λ_0 . Hence for each $x \in \{0, 1, \dots\}$ we have

$$\begin{aligned} P_{r_n, b_n}(X = x) &= EP(X = x|\Lambda_n) = E\left\{\frac{\Lambda_n^x}{x!} \exp(-\Lambda_n)\right\} \\ (3) \quad &\rightarrow \frac{\lambda_0^x}{x!} \exp(-\lambda_0) \end{aligned}$$

by the Helly-Bray theorem applied to the bounded continuous function $g(v) = v^x \exp(-v)$ on $[0, \infty)$ and $\Lambda_n \rightarrow_d \lambda_0$. This implies that $X_n \rightarrow_d X_0 \sim \text{Poisson}(\lambda_0)$. The convergence in (3) can also be seen directly from the results in (a) as follows:

$$\begin{aligned} P_{r_n, b_n}(X = x) &= \frac{\Gamma(r_n + x)}{x! \Gamma(r_n)} \left(\frac{b_n}{1 + b_n}\right)^{r_n} \frac{1}{(1 + b_n)^x} \\ (4) \quad &= \frac{1}{x!} \frac{\Gamma(r_n + x)}{\Gamma(r_n)(1 + b_n)^x} \left(\frac{1}{1 + (1/b_n)}\right)^{r_n} \end{aligned}$$

where

$$\left(\frac{1}{1 + (1/b_n)}\right)^{r_n} = \frac{1}{(1 + \frac{\lambda_0}{\sqrt{n}})^{\sqrt{n}}} \rightarrow \frac{1}{\exp(\lambda_0)} = \exp(-\lambda_0)$$

and, using Stirling's formula, $\Gamma(r+1) \sim (\sqrt{2\pi r}(r/e))^r$ as $r \rightarrow \infty$,

$$\begin{aligned} \frac{\Gamma(r_n + x)}{\Gamma(r_n)(1 + b_n)^x} &\sim \frac{\sqrt{2\pi(r_n + x - 1)} \left(\frac{r_n + x - 1}{e}\right)^{r_n + x - 1}}{\sqrt{2\pi(r_n - 1)} \left(\frac{r_n - 1}{e}\right)^{r_n - 1}} \frac{\lambda_0^x}{(\sqrt{n})^x} \\ &= \sqrt{\frac{r_n + x - 1}{r_n - 1}} \left(\frac{r_n + x - 1}{r_n - 1}\right)^{r_n - 1} e^{-x} \left(\frac{r_n + x - 1}{\sqrt{n} - 1}\right)^x \cdot \lambda_0^x \\ &\rightarrow 1 \cdot e^x e^{-x} \cdot 1 \cdot \lambda_0^x = \lambda_0^x. \end{aligned}$$

Thus the expression on the right side in (4) converges to the Poisson probability

$$\frac{1}{x!} \lambda_0^x \exp(-\lambda_0).$$

(d) First write

$$(5) \quad \sqrt{n} \left(\frac{S_n^2}{\bar{X}_n} - 1 \right) = \sqrt{n} \left(\frac{S_n^2}{\bar{X}_n} - \frac{\sigma_n^2}{\mu_n} \right) + \sqrt{n} \left(\frac{\sigma_n^2}{\mu_n} - 1 \right)$$

where

$$\begin{aligned} \sigma_n^2 &= \text{Var}_n(X) = \lambda_0 + n^{-1/2} \lambda_0^2, \\ \mu_n &= E_n(X) = \lambda_0. \end{aligned}$$

Thus

$$(6) \quad \sqrt{n} \left(\frac{\sigma_n^2}{\mu_n} - 1 \right) = \sqrt{n}(1 + n^{-1/2} \lambda_0 - 1) = \lambda_0,$$

and if we show that

$$(7) \quad \sqrt{n} \left(\frac{S_n^2}{\bar{X}_n} - \frac{\sigma_n^2}{\mu_n} \right) \rightarrow_d \sqrt{2}Z \sim N(0, 2),$$

where $Z \sim N(0, 1)$, then it follows from (5), (6), (7), that

$$T_n \equiv \sqrt{n} \left(\frac{S_n^2}{\bar{X}_n} - 1 \right) \rightarrow_d \sqrt{2}Z + \lambda_0 \sim N(\lambda_0, 2).$$

This yields an approximation to the power of the test derived in problem 2 under this sequence of Negative binomial alternatives:

$$\begin{aligned} \text{Power}(\text{NegBin}(r_n, b_n)) &= P_{r_n, b_n}(T_n > \sqrt{2}z_\alpha) \rightarrow P(\sqrt{2}Z + \lambda_0 > \sqrt{2}z_\alpha) \\ &= P(Z > z_\alpha - \lambda_0/\sqrt{2}) = 1 - \Phi(z_\alpha - \lambda_0/\sqrt{2}). \end{aligned}$$

(This is the power of an ad-hoc test based on comparison of two different moments of the Poisson distribution. What is the limiting power of an optimal test for distinguishing this sequence of negative binomial alternatives? We will return to this later!)

To show that (7) holds, it suffices, by the delta method, to show that

$$(8) \quad \sqrt{n} \begin{pmatrix} \bar{X}_n - \mu_n \\ S_n^2 - \sigma_n^2 \end{pmatrix} \rightarrow_d N_2(0, \Sigma)$$

where

$$\Sigma = \begin{pmatrix} \lambda_0 & \lambda_0 \\ \lambda_0 & \lambda_0 + 2\lambda_0^2 \end{pmatrix}.$$

The rest of the proof is concerned (only) with showing that (8) holds. As before,

$$\begin{aligned} \sqrt{n} \begin{pmatrix} \bar{X}_n - \mu_n \\ S_n^2 - \sigma_n^2 \end{pmatrix} &= \sqrt{n} \begin{pmatrix} \bar{X}_n - \mu_n \\ \bar{Y}_n \end{pmatrix} + o_p(1) \\ &\equiv Z_n + o_p(1) \end{aligned}$$

where $Y_i = (X_i - \mu_n)^2 - \sigma_n^2$, so it suffices to show that $Z_n \rightarrow_d N_2(0, \Sigma)$.

To do this, let $a \in \mathbb{R}^2$, and consider

$$\begin{aligned} a'Z_n &= a_1\sqrt{n}(\bar{X}_n - \mu_n) + a_2\sqrt{n}\bar{Y}_n \\ &= \sum_{i=1}^n \{a_1n^{-1/2}(X_i - \mu_n) + a_2n^{-1/2}[(X_i - \mu_n)^2 - \sigma_n^2]\} \\ &= \sum_{i=1}^n X_{ni} \end{aligned}$$

where $\mu_{ni} = EX_{ni} = 0$ and

$$\sigma_{ni}^2 = \text{Var}(X_{ni}) = a_1^2n^{-1}\text{Var}_n(X_1) + 2a_1a_2n^{-1}E_n(X_1 - \mu_n)^3 + a_2^2n^{-1}\{E_n(X_1 - \mu_n)^4 - \sigma_n^4\},$$

so that $\mu_n = \sum_1^n \mu_{ni} = 0$, and

$$\begin{aligned} \sigma_n^2 &= \sum_{i=1}^n \sigma_{ni}^2 = a_1^2\text{Var}_n(X_1) + 2a_1a_2E_n(X_1 - \mu_n)^3 + a_2^2\{E_n(X_1 - \mu_n)^4 - \sigma_n^4\} \\ (9) \quad &\rightarrow a'\Sigma a > 0 \end{aligned}$$

since, using the results of (b)

$$\begin{aligned} \text{Var}_n(X) &= \lambda_0 + \lambda_0^2/\sqrt{n} \rightarrow \lambda_0, \\ E_n(X_1 - \mu_n)^3 &= E_nE[(X_1 - \mu_n)^3|\Lambda_n] \rightarrow \lambda_0, \\ E_n(X_1 - \mu_n)^4 - \sigma_n^4 &\rightarrow \lambda_0 + 2\lambda_0^2. \end{aligned}$$

Now we verify the hypothesis of the Liapunov CLT: first,

$$\begin{aligned} \gamma_{ni} &\equiv E|X_{ni}|^3 \\ &\leq 2^2\{E_n|a_1n^{-1/2}(X_i - \mu_n)|^3 + E_n|a_2n^{-1/2}Y_i|^3\} \\ &\leq 2^2\{|a_1|^3n^{-3/2}E_n|X_1 - \mu_n|^3 + |a_2|^3n^{-3/2}|Y_1|^3\}. \end{aligned}$$

Therefore

$$\begin{aligned} \gamma_n &\equiv \sum_{i=1}^n \gamma_{ni} \\ &\leq 2^2\{|a_1|^3n^{-1/2}E_n|X_1 - \mu_n|^3 + |a_2|^3n^{-1/2}E_n|Y_1|^3\}. \end{aligned}$$

Thus we see that $\gamma_n/\sigma_n^3 \rightarrow 0$ if both $\limsup_{n \rightarrow \infty} E_n|X_1 - \mu_n|^3 < \infty$ and $\limsup_{n \rightarrow \infty} E_n|(X_1 - \mu_n)^2 - \sigma_n^2|^3 < \infty$. By applying Minkowski's inequality (the triangle inequality for the L_3 norm), it is clear that this will hold if the same is true for $E_n|X_1|^3$ and $E_n|X_1|^6$. But note that for $x \in \{0, 1, \dots\}$, any positive integer m and number $t > 0$, $t^m|x|^m/m! = (tx)^m/m! \leq \exp(tx)$, so $E_n|X_1|^m \leq (m!/t^m)E_n \exp(tX_1)$, and hence it suffices to show that $\limsup_{n \rightarrow \infty} E_n \exp(tX_1) < \infty$. But since the moment generating function of $U \sim \text{Poisson}(\lambda)$ is $E \exp(tU) =$

$\exp(\lambda(e^t - 1))$

$$\begin{aligned}
E_n \exp(tX_1) &= E_n E[\exp(tX_1)|\Lambda_n] = E_n \exp(\Lambda_n(e^t - 1)) \\
&= \int_0^\infty \exp(\lambda(e^t - 1)) \frac{b_n(b_n\lambda)^{r_n-1}}{\Gamma(r_n)} \exp(-b_n\lambda) d\lambda \\
&= \int_0^\infty \frac{b_n(b_n\lambda)^{r_n-1}}{\Gamma(r_n)} \exp(-(b_n - (e^t - 1))\lambda) d\lambda \\
&= \left(\frac{b_n}{b_n - (e^t - 1)} \right)^{r_n} \\
&= \left(1 - \frac{e^t - 1}{b_n} \right)^{-r_n} \\
&= \left(1 - \frac{\lambda_0(e^t - 1)}{\sqrt{n}} \right)^{-\sqrt{n}} \\
&\rightarrow \exp(\lambda_0(e^t - 1)).
\end{aligned}$$

(Note that this is, in fact, the moment generating function of a Poisson random variable with parameter λ_0 .) Thus $\gamma_n/\sigma_n^3 \rightarrow 0$ holds, and via the Liapunov CLT we find that $a'Z_n/\sigma_n \rightarrow_d N(0, 1)$. In view of (9) this yields $a'Z_n \rightarrow_d a'Z \sim N(0, a'\Sigma a)$, and by the Cramér-Wold device we conclude $Z_n \rightarrow_d Z \sim N_2(0, \Sigma)$.

4. Ferguson, ACILST, problem 1, page 54: Find the asymptotic distribution of the estimate of the regression coefficient $\hat{\beta} = S_{XY}/S_X^2$ when sampling from a bivariate distribution (with finite fourth moments). What is its asymptotic variance when sampling from a bivariate normal distribution?

Solution: (From Ferguson, page 192, modified slightly.) Let $\mu_{jk} \equiv E(X - EX)^j(Y - EY)^k$. Thus $\mu_{11} = \sigma_{XY}$, $\mu_{20} = \sigma_X^2$. By the same result we derived in treating the correlation coefficient,

$$\sqrt{n} \begin{pmatrix} S_{XY} - \sigma_{XY} \\ S_X^2 - \sigma_X^2 \end{pmatrix} \rightarrow_d Z \sim N_2(0, \Sigma)$$

where

$$\Sigma = \begin{pmatrix} \mu_{22} - \mu_{11}^2 & \mu_{31} - \mu_{20}\mu_{11} \\ \mu_{31} - \mu_{20}\mu_{11} & \mu_{40} - \mu_{20}^2 \end{pmatrix}.$$

Taking $g(t, u) = t/u$ we find $g(S_{XY}, S_X^2) = S_{XY}/S_X^2 \equiv \hat{\beta}_n$, $g(\sigma_{XY}, \sigma_X^2) = \sigma_{XY}/\sigma_X^2 \equiv \beta$, $g'(t, u) = (1/u, -t/u^2)$,

$$g'(\sigma_{XY}, \sigma_X^2) = (1, -\beta)/\sigma_X^2 = \frac{1}{\mu_{20}} \begin{pmatrix} 1 \\ -\mu_{11} \\ \mu_{20} \end{pmatrix},$$

so

$$\begin{aligned}
\sqrt{n}(\hat{\beta}_n - \beta) &= \sqrt{n}(g(S_{XY}, S_X^2) - g(\sigma_{XY}, \sigma_X^2)) \\
&\rightarrow_d g'(\sigma_{XY}, \sigma_X^2)Z \sim N(0, \dot{g}^T \Sigma \dot{g}) \\
&= N(0, [\mu_{22}\mu_{20}^2 - 2\mu_{31}\mu_{20}\mu_{11} + \mu_{40}\mu_{11}^2]/\mu_{20}^4).
\end{aligned}$$

When the (X_i, Y_i) 's are bivariate normal, then $\mu_{40} = 3\Sigma_X^4$, $\mu_{31} = 3\rho\sigma_X^2\sigma_Y$, $\mu_{22} = (1 + 2\rho^2)\sigma_X^2\sigma_Y^2$, and $\mu_{11} = \rho\sigma_X\sigma_Y$. Therefore the asymptotic variance becomes, in this case,

$$[(1 + 2\rho^2)\sigma_X^6\sigma_Y^2 - 6\rho^2\sigma_X^6\sigma_Y^2 + 3\rho^2\rho_X^6\sigma_Y^2] / \sigma_X^8 = (1 - \rho^2)\sigma_Y^2 / \sigma_X^2.$$

5. Suppose that $X_{n,1}, \dots, X_{n,n}$ are independent Bernoulli($p_{n,1}$), \dots , Bernoulli($p_{n,n}$) respectively. Let $T_n = X_{n,1} + \dots + X_{n,n}$, $\mu_n = \sum_{i=1}^n p_{n,i}$, and $\sigma_n^2 \equiv \sum_{i=1}^n p_{n,i}(1 - p_{n,i})$.
 (a) Use the Liapunov central limit theorem to show that if $\sigma_n^2 \rightarrow \infty$, then $(T_n - \mu_n) / \sigma_n \rightarrow_d N(0, 1)$.

(b) Show that the key condition of the Liapunov CLT implies the Lindeberg condition.

(Note that part (a) of this problem is connected to the Optional bonus problem 7 of problem set 3. For better bounds in the Poisson approximation, see the Introduction of "Poisson Approximation" by Barbour, Holst, and Janson.)

Solution: (a) We compute $\mu_{ni} \equiv EX_{ni} = p_{ni}$, $\sigma_{ni}^2 = \text{Var}(X_{ni}) = p_{ni}(1 - p_{ni})$, and

$$\begin{aligned} \gamma_{ni} &\equiv E|X_{ni} - p_{ni}|^3 = p_{ni}|1 - p_{ni}|^3 + (1 - p_{ni})|0 - p_{ni}|^3 \\ &= p_{ni}(1 - p_{ni})\{(1 - p_{ni})^2 + p_{ni}^2\} \\ &\leq p_{ni}(1 - p_{ni}) = \sigma_{ni}^2. \end{aligned}$$

Thus $\mu_n \equiv \sum_1^n \mu_{ni}$, $\sigma_n^2 = \sum_1^n p_{ni}(1 - p_{ni})$, and

$$\gamma_n \equiv \sum_1^n \gamma_{ni} \leq \sum_1^n \sigma_{ni}^2 = \sigma_n^2.$$

It follows that $\gamma_n / \sigma_n^3 \leq \sigma_n^2 / \sigma_n^3 = 1 / \sigma_n \rightarrow 0$ if $\sigma_n^2 \rightarrow \infty$. So the Liapunov CLT yields $(T_n - \mu_n) / \sigma_n \rightarrow_d N(0, 1)$ under the condition $\sigma_n^2 = \sum_1^n p_{ni}(1 - p_{ni}) \rightarrow \infty$.

(b) Note that

$$|w|^r 1_{[|w|>c]} \leq |w|^r \frac{|w|}{c} 1_{[|w|>c]} \leq \frac{|w|^{r+1}}{c}.$$

Using this with $r = 2$, $c = \epsilon\sigma_n$, $w = X_{ni}$ yields

$$\begin{aligned} \frac{1}{\sigma_n^2} \sum_{i=1}^n E\{|X_{ni}|^2 1_{[|X_{ni}|>\epsilon\sigma_n]}\} &\leq \frac{1}{\sigma_n^2} \sum_{i=1}^n E\left\{|X_{ni}|^2 \frac{|X_{ni}|}{\epsilon\sigma_n} 1_{[|X_{ni}|>\epsilon\sigma_n]}\right\} \\ &\leq \frac{1}{\sigma_n^2} \sum_{i=1}^n \frac{E\{|X_{ni}|^3\}}{\epsilon\sigma_n} \\ &= \frac{1}{\epsilon\sigma_n^3} \sum_{i=1}^n E|X_{ni}|^3 = \frac{1}{\epsilon} \frac{\gamma_n}{\sigma_n^3} \end{aligned}$$

for X_{ni} 's with $EX_{ni} = 0$. Thus $\gamma_n / \sigma_n^3 \rightarrow 0$ implies that the Lindeberg condition holds.

6. **Optional bonus problem.** Suppose that X_1, X_2, \dots are i.i.d. positive random variables, and define $\bar{X}_n \equiv n^{-1} \sum_{i=1}^n X_i$, $H_n \equiv 1 / (n^{-1} \sum_{i=1}^n (1/X_i))$, and $G_n \equiv \{\prod_{i=1}^n X_i\}^{1/n}$ to be the *arithmetic*, *harmonic*, and *geometric* means respectively.

We know that $\bar{X}_n \rightarrow_{a.s.} E(X_1) = \mu$ if and only if $E|X_1| < \infty$.

(a) Use the SLLN together with appropriate additional hypotheses to show that $H_n \rightarrow_{a.s.} 1/\{E(1/X_1)\} \equiv h$, and $G_n \rightarrow_{a.s.} \exp(E\{\log X_1\}) \equiv g$.

(c) Use the multivariate CLT and the delta method to find the joint limiting distribution of $\sqrt{n}(\bar{X}_n - \mu, H_n - h, G_n - g)$. You will need to impose or assume additional moment conditions to be able to prove this. Specify these additional assumptions carefully.

(d) Suppose that $X_i \sim \text{Gamma}(r, \lambda)$ with $r > 0$. For what values of r are the hypotheses you imposed in (c) satisfied? Compute the covariance of the limiting distribution in (c) as explicitly as you can in this case.

(e) Use the result in (c) to show that $\sqrt{n}(G_n/\bar{X}_n - g/\mu) \rightarrow_d N(0, V^2)$ and compute V^2 explicitly when $X_i \sim \text{Gamma}(r, \lambda)$ with r satisfying the conditions you found in (d).