Statistics 581, Problem Set 9

Wellner; 11/29/2006

Reading: Chapter 4, Sections 1-2;
Ferguson, ACLST, Chapters 18- 20, pages 119-125, 133-139; Chapter 22, pages 144-150;
Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.
Due: Wednesday, December 6, 2006.

1. Lehmann and Casella, TPE, problem 6.3.22, page 503, reworded as follows. (In other words, prove (vi) of theorem 1.2, pages 5-6, chapter 4 notes). Suppose that $X_1, \ldots, X_n$ are i.i.d. with density p_θ, $\theta \in \Theta \subset R^k$, satisfying the hypotheses of theorem 4.1, page 463 (the Cramér conditions given in (A) - (D) on pages 462-463). Show that the following Local Asymptotic Normality (LAN) result holds for the (local) log- likelihood ratios: with

$$L_n(\theta) \equiv \log(\prod_{i=1}^{n} p_\theta(X_i)) = \sum_{i=1}^{n} \log p_\theta(X_i),$$

for a fixed $\theta_0 \in \Theta$,

$$\begin{aligned} L_n(\theta_0 + n^{-1/2}\underline{t}) - L_n(\theta_0) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underline{t}^T \dot{\underline{l}}_\theta(X_i) - \frac{1}{2}\underline{t}^T I(\theta_0)\underline{t} + o_p(1) \\ &\to_d N(0, t^T I(\theta_0)t) - \frac{1}{2}t^T I(\theta_0)t \\ &\stackrel{d}{=} N(-\sigma^2/2, \sigma^2) \end{aligned}$$

under P_{θ_0} where $\sigma^2 \equiv \underline{t}^T I(\theta_0)\underline{t}$. (The convergence in the last display actually holds under the considerably weaker hypothesis of Hellinger differentiability of p_θ at θ_0, as stated in Corollary 3 of section 3.3, page 28, of the Chapter 3 notes.)

2. (a) Exercise 2.1.6, page 10, chapter 2 notes.
(b) Exercise 2.1.7, page 10, chapter 2 notes.

3. Consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities

$$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}\exp(-(\frac{x}{\alpha})^\beta)1_{[0,\infty)}(x)$$

where $\theta \equiv (\alpha, \beta) \in (0, \infty) \times (0, \infty) \subset R^2$. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with density function p_θ.
(a) If $X \sim P_\theta \in \mathcal{P}$, show that the distributions of $\log X$ form a location and scale family from a Gumbel (extreme value) density on R.
(b) Use the result of A to construct method of moments estimators or quantile based estimators $\overline{\theta}_n$ of $\theta = (\alpha, \beta)$.
(c) Show that the method of moments or quantile estimators $\overline{\theta}_n$ of θ are asymptotically normal, and find the asymptotic distribution; i.e. show that

$$\sqrt{n}(\overline{\theta}_n - \theta) \to_d N_2(0, \Sigma) \qquad \text{for some} \qquad \Sigma.$$

[We will use these estimators as “starting points” approximate (or one-step) maximum likelihood estimators in the next problem .]

4. (Problem #3, continued).
(a) Does a maximum likelihood estimate of $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ exist? Is it unique? (See Lehmann and Casella, Example 6.1, page 468.)
(b) Compute an approximate (one - step) maximum likelihood estimate $\check{\theta}$ of θ using the method of moment (or quantile) estimators $\overline{\theta}_n$ as the preliminary estimators based on the following data (with $n = 12$):

$$1,\ 1,\ 2,\ 3,\ 12,\ 25,\ 46,\ 56,\ 68,\ 109,\ 323,\ 417.$$

[These are failure times in seconds for “breakdown” of an insulating fluid between two electrodes subject to a voltage of 40 kV. – from Nelson, *Applied Life Data Analysis*, page 252.]
(c) Compute the maximum likelihood estimator $\hat{\theta}_n$, and compare it with the one step estimator computed in E.

5. (a) Ferguson, ACILST, problem 17.2, page 117.
(b) Do our hypotheses A0-A2 hold in this example?
(c) Compute $K(P_{\theta_0}, P_\theta)$ where P_θ has density as given in this problem.
(d) Do our hypotheses A3 and A4 hold in this example? Why or why not?
(e) Does there exist an estimator $\overline{\theta}_n$ of θ which is $n^{1/2}-$consistent?

6. **Optional bonus problem:** Ferguson, ACILST, problem 17.3, page 118.