

Statistics 581, Problem Set 4

Wellner; 10/18/2006

Reading: Course Notes, Chapter 2, sections 3-6; Ferguson, ACILST pages 44 - 66.

Due: Wednesday, October 225, 2006.

1. Suppose that $\underline{N}_n \sim \text{Mult}_k(n, \underline{p})$ and $\hat{p} = \underline{N}_n/n$. Suppose that the true \underline{p} is $\underline{p}_n = \underline{p}_0 + n^{-1/2}\underline{c}$ where $\underline{1}^T \underline{c} = 0$. Use the Cramér - Wold device together with either the Liapunov or the Lindeberg-Feller CLT to show that

$$\underline{Z}_n = \left(\frac{N_{n,1} - np_{n,1}}{\sqrt{np_{0,1}}}, \dots, \frac{N_{nk} - np_{n,k}}{\sqrt{np_{0,k}}} \right)$$

satisfies $\underline{Z}_n \rightarrow_d \underline{Z}$ where $\underline{Z} \sim N_k(0, I - \sqrt{p_0}\sqrt{p_0}^T)$. (It therefore follows, as outlined in class, that the chi-square statistic $Q_n \rightarrow_d \chi_{k-1}^2(\delta)$ with $\delta = \sum_{j=1}^k c_j^2/p_{0,j}$ under the local alternative \underline{p}_n .)

2. Ferguson, ACILST, problem 5, page 50: (The Poisson dispersion test). A standard test of the hypothesis H_0 that a distribution is Poisson(λ) for some λ is to reject H_0 if the ratio of the sample variance to the sample mean, S_n^2/\bar{X}_n , is too large. This test is good against alternatives whose variance is greater than the mean, such as the negative binomial distribution or any other mixture of Poisson distributions.
 - (a) Find the asymptotic distribution of S_n^2/\bar{X}_n for general distributions.
 - (b) Find the asymptotic distribution of S_n^2/\bar{X}_n under H_0 and show that it is independent of λ .

3. (Continuation of problem 2 above.) Suppose that $(X|\Lambda) \sim \text{Poisson}(\Lambda)$ where $\Lambda \sim \Gamma(r, b)$ with density $b^r \lambda^{r-1} \exp(-b\lambda)/\Gamma(r)$ for some $r > 0$ and $b > 0$.
 - (a) Show that the marginal distribution of X is Negative Binomial $(b/(1+b), r)$ with density (probability mass function)

$$P_{r,b}(X = x) = \frac{\Gamma(r+x)}{x!\Gamma(r)} \left(\frac{b}{1+b} \right)^r \frac{1}{(1+b)^x}$$

for $x = 0, 1, \dots$

(b) Show that $E(X) = r/b$ and $\text{Var}(X) = (r/b) + r/b^2 > (r/b) = E(X)$, and hence if $b \equiv b_n = \sqrt{n}/\lambda_0$ and $r \equiv r_n = \sqrt{n}$, we have, letting E_n and Var_n denote expectation and variance under (r_n, b_n) , $E_n(X) \rightarrow \lambda_0$ and $\text{Var}_n(X) \rightarrow \lambda_0$, while $\sqrt{n}(\text{Var}_n(X) - \lambda_0) \rightarrow \lambda_0^2$. (Hint: Use our results for computing the mean and variance conditionally on another random variable.)

(c) Show that if $X_n \sim \text{Negative Binomial}(b_n/(1+b_n), r_n)$ with b_n and r_n as in (b), then $X_n \rightarrow_d X_0 \sim \text{Poisson}(\lambda_0)$.

(d) Now suppose that $X_{ni} \sim \text{Negative Binomial}(b_n/(1+b_n), r_n)$ for $i = 1, \dots, n$ are independent with b_n and r_n as in (b). Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_{ni}$ and $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_{ni} - \bar{X}_n)^2$ as in problem 2. Use the results of (b) to show that under this family of local alternatives to the Poisson distribution we have

$$\sqrt{n}(S_n^2/\bar{X}_n - 1) \rightarrow_d N(c, 2)$$

for some $c \neq 0$ and find c . Use this to approximate the Power of the test in problem 2 for this particular sequence of alternatives.

4. Ferguson, ACILST, problem 1, page 54: Find the asymptotic distribution of the estimate of the regression coefficient $\hat{\beta} = S_{XY}/S_X^2$ when sampling from a bivariate distribution (with finite fourth moments). What is its asymptotic variance when sampling from a bivariate normal distribution?
5. Suppose that $X_{n,1}, \dots, X_{n,n}$ are independent Bernoulli($p_{n,1}$), \dots , Bernoulli($p_{n,n}$) respectively. Let $T_n = X_{n,1} + \dots + X_{n,n}$, $\mu_n = \sum_{i=1}^n p_{n,i}$, and $\sigma_n^2 \equiv \sum_{i=1}^n p_{n,i}(1-p_{n,i})$.
 - (a) Use the Liapunov central limit theorem to show that if $\sigma_n^2 \rightarrow \infty$, then $(T_n - \mu_n)/\sigma_n \rightarrow_d N(0, 1)$.
 - (b) Show that the key condition of the Liapunov CLT implies the Lindeberg condition.
(Note that part (a) of this problem is connected to the Optional bonus problem 7 of problem set 3. For better bounds in the Poisson approximation, see the Introduction of “Poisson Approximation” by Barbour, Holst, and Janson.)
6. **Optional bonus problem.** Suppose that X_1, X_2, \dots are i.i.d. positive random variables, and define $\bar{X}_n \equiv n^{-1} \sum_{i=1}^n X_i$, $H_n \equiv 1/(n^{-1} \sum_{i=1}^n (1/X_i))$, and $G_n \equiv \{\prod_{i=1}^n X_i\}^{1/n}$ to be the *arithmetic*, *harmonic*, and *geometric* means respectively. We know that $\bar{X}_n \rightarrow_{a.s.} E(X_1) = \mu$ if and only if $E|X_1| < \infty$.
 - (a) Use the SLLN together with appropriate additional hypotheses to show that $H_n \rightarrow_{a.s.} 1/\{E(1/X_1)\} \equiv h$, and $G_n \rightarrow_{a.s.} \exp(E\{\log X_1\}) \equiv g$.
 - (c) Use the multivariate CLT and the delta method to find the joint limiting distribution of $\sqrt{n}(\bar{X}_n - \mu, H_n - h, G_n - g)$. You will need to impose or assume additional moment conditions to be able to prove this. Specify these additional assumptions carefully.
 - (d) Suppose that $X_i \sim \text{Gamma}(r, \lambda)$ with $r > 0$. For what values of r are the hypotheses you imposed in (c) satisfied? Compute the covariance of the limiting distribution in (c) as explicitly as you can in this case.
 - (e) Use the result in (c) to show that $\sqrt{n}(G_n/\bar{X}_n - g/\mu) \rightarrow_d N(0, V^2)$ and compute V^2 explicitly when $X_i \sim \text{Gamma}(r, \lambda)$ with r satisfying the conditions you found in (d).