

Statistics 581, Problem Set 2

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Reading: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 1-25.

Due: Wednesday, October 11, 2006.

- (a) If $W \sim \chi_2^2 = \text{Gamma}(2/2, 1/2) = \text{Gamma}(1, 1/2)$, find the density function f_W , distribution function F_W , and inverse distribution function F_W^{-1} explicitly.
(b) Suppose that $(X, Y) \sim N_2(0, I)$. Show that R and Θ defined by $R^2 = X^2 + Y^2$ and $\Theta = \arctan(Y/X)$ are independent random variables with $R^2 \sim \chi_2^2$ and $\Theta \sim \text{Uniform}(0, 2\pi)$.
(c) Use the results of (a) and (b) to show (using Theorem 2.3.4) how to use two independent $\text{Uniform}(0, 1)$ random variables U and V to generate two standard normal random variables.
- Suppose that Y is a random variable with $E(Y^2) < \infty$; i.e. $Y \in L_2(P)$.
 - Show that

$$\text{Var}(Y) = E\{\text{Var}(Y|X)\} + \text{Var}\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}.$$

- Interpret (a) geometrically.
- Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $\text{Var}(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

- Suppose that $X \sim F$ on $R^+ \equiv [0, \infty)$, $Y \sim G$ on R^+ , and X and Y are independent random variables. Let $Z = \min\{X, Y\} = X \wedge Y$ and $\Delta = 1\{X \leq Y\}$. (This is *right-censored data*: if we view X as a survival time, and Y as a censoring time, then $Z = X$ when $X \leq Y$, but $Z = Y$ when $X > Y$.)
 - Find the joint distribution of (Z, Δ) by computing the two sub-distribution functions $F_{uc}(z) \equiv P(Z \leq z, \Delta = 1)$ and $F_c(t) \equiv P(Z \leq z, \Delta = 0)$.
 - If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, show that Z and Δ are independent.
 - Repeat (a) when Z is replaced by $\tilde{Z} = Y$, but Δ is still $\Delta = 1\{X \leq Y\}$. Are \tilde{Z} and Δ independent when $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$?

4. (i) Ferguson, ACILST, #6, page 12.
(ii) Show that the problem #6 in Ferguson continues to hold if f_n and g are densities with respect to an arbitrary fixed dominating measure μ .
(iii) Give examples of densities f_n and g satisfying $f_n(x) \rightarrow g(x)$ for all x in the two cases $\mu = \text{Lebesgue measure on } R$; $\mu = \text{counting measure on } \{0, 1, \dots\}$.
5. Suppose that X_1, X_2, \dots are iid Exponential(λ). Let $M_n \equiv \min_{1 \leq i \leq n} X_i$ and $T_n \equiv \max_{-1 \leq i \leq n} X_i$.
(a) Show that $nM_n \stackrel{d}{=} \text{exponential}(\lambda)$.
(b) Show that $T_n - (1/\lambda) \log n \rightarrow_d (1/\lambda)T$ where T has the double exponential extreme value distribution function given by $P(T \leq x) = \exp(-\exp(-x))$.
(c) Now suppose that X_1, \dots, X_n are iid distribution function F satisfying $0 < F'(0) < \infty$; here $F'(0)$ is the right-derivative of F at 0:

$$\lim_{x \searrow 0} \frac{F(x) - F(0)}{x} = F'(0).$$

Show that $nM_n \rightarrow_d \text{exponential}(F'(0))$.

6. Let $X_n \sim \chi_n^2$ for $n \geq 1$.
(a) Show that $\sqrt{n/2}(n^{-1}X_n - 1) \rightarrow_d N(0, 1)$.
(b) Is the asymptotic distribution of $Y_n \equiv (1 - 1/n)X_n$ the same as that of X_n ? That is, does $\sqrt{n/2}(n^{-1}Y_n - 1) \rightarrow_d N(0, 1)$? How about that of $T_n \equiv (1 - 1/\sqrt{n})X_n$?
(c) Show that

$$\frac{\sqrt{n} \left\{ \left(\frac{X_n}{n} \right)^{1/3} - \left(1 - \frac{2}{9n} \right) \right\}}{\sqrt{2/9}} \rightarrow_d N(0, 1)$$

as $n \rightarrow \infty$. [In fact, this convergence occurs very rapidly; this is the ‘‘Wilson - Hilferty’’ transformation of a Chi-square random variable. This is shown very clearly by normal probability plots comparing the approximations in (a) and (c).]