Statistics 581, Final Exam
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This exam is to be taken without the use of any books or notes.

1. (40 points) **Define** the following terms. In each case, provide an appropriate (brief) context for your definition.
   (a) The *Kullback - Leibler* divergence (or information) between a probability measure P and another (sub-)probability measure Q on the same measurable space $(\mathcal{X}, \mathcal{A})$.
   (b) The *Hellinger distance* between two probability measures P and Q on a measurable space $(\mathcal{X}, \mathcal{A})$.
   (c) The vector of score functions for a sample of size one in a regular parametric model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}^d$.
   (d) The *information matrix* for a sample of size one in a regular parametric model $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}^d$.
   (e) An *asymptotically linear estimator* with influence function ψ.
   (f) The efficient influence function $\tilde{\mathbf{l}}_\nu$ for a differentiable parameter $q(\theta) = \nu(P_\theta)$ in a regular parametric model $\mathcal{P}$.

2. (24 points) **State** the following results:
   (a) The multiparameter Cramér - Rao inequality (for an unbiased estimator) $T = T(\underline{X})$ of a real-valued parameter $q(\theta)$.
   (b) LAN (Local Asymptotic Normality) of the local - log likelihood ratios for a regular parametric model satisfying the Cramér hypotheses (continuous third derivatives with integrable bounds).
   (c) The asymptotic behavior of the likelihood ratio statistic $2 \log \lambda_n$ for testing a simple null hypothesis $\theta = \theta_0$ versus $\theta \neq \theta_0$ under a fixed alternative P_θ with $\theta \neq \theta_0$.
   (d) A limiting distribution result for $\sqrt{n}(\mathbb{F}_n^{-1}(t_1) - F^{-1}(t_1), \ldots, \mathbb{F}_n^{-1}(t_k) - F^{-1}(t_k))$ for fixed numbers $0 < t_1 < \ldots < t_k < 1$.

3. (40 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. $P_{\theta_0} \in \mathcal{P}$ where $\mathcal{P} = \{P_\theta : \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty)\}$ is the Weibull family: thus

$$p_\theta(x) = (\beta/\alpha)(x/\alpha)^{\beta-1} \exp(-(x/\alpha)^\beta) 1_{(0,\infty)}(x)$$

   for $\alpha > 0$, $\beta > 0$. Suppose that we are interested in estimating $\nu(P_\theta) = \theta_1 = \alpha$, and that T_n is an asymptotically linear estimator of $\nu(P_\theta)$ with influence function ψ (that is, T_n could be a method of moments or quantile based estimator of $\theta_1 = \alpha$.
   A. What is the efficient influence function $\tilde{\mathbf{l}}_1$ for estimation of θ_1 (in terms of

the scores and information quantities)? What is the efficient score $\mathbf{l}_1^*$ function for estimation of θ_1?
B. Draw a picture showing the relationship of the influence function ψ of T_n to $\tilde{\mathbf{l}}_1$ and the tangent space $\dot{\mathcal{P}} = [\dot{\mathbf{l}}]$ of $\mathcal{P}$ (at θ_0).
C. The picture you drew in B shows that

$$\psi - \tilde{\mathbf{l}}_1 \perp \dot{\mathcal{P}} \qquad \text{in } \; L_2(P_{\theta_0}). \tag{1}$$

Show that (1) is equivalent to the pair of equations

$$E_0(\psi \dot{\mathbf{l}}_1) = 1, \qquad E_0(\psi \dot{\mathbf{l}}_2) = 0. \tag{2}$$

(That is, (1) implies both of the equalities in (2), while (2) implies (1).)
D. Draw a picture relating the *score* function $\dot{\mathbf{l}}_1 = \dot{\mathbf{l}}_\alpha$ to the efficient score function $\mathbf{l}_1^*$ and the "nuisance parameter" score function $\dot{\mathbf{l}}_2 = \dot{\mathbf{l}}_\beta$.
E. Relate the information bound $I_{11\cdot 2}^{-1}$ to the efficient influence function $\tilde{\mathbf{l}}_1$ and the efficient socre function $\mathbf{l}_1^*$ in parts B and D.

4. (40 points). This is a continuation of problem 3 above; all the questions below concern the Weibull model with $\theta = (\alpha, \beta)$ as in problem 3.
A. Consider testing $H : \theta = \theta_0$ versus $K : \theta \neq \theta_0$. Describe the test statistics we have available for testing H versus K and give their asymptotic distributions under the null hypothesis H.
B. Describe the limiting distributions of the test statistics in A under local alternatives of the form $\theta_0 + tn^{-1/2}$.
C. Describe the limiting behavior of the test statistics in A under a fixed alternative $\theta \neq \theta_0$.
D. Now consider testing $H : \beta = 1$ versus $K : \beta \neq 1$. Discuss the test statistics we have available for testing H versus K and give their asymptotic distributions under the null hypothesis H.
E. What is the limiting behavior of the test statistics in D under local alternatives of the form $\theta_n = \theta_0 + tn^{-1/2}$ with $\theta_0 = (\alpha, \beta_0) = (\alpha, 1) \in H$?
F. Write out the test statistics in part D as explicitly as possible for the given Weibull model. Recall that the information matrix for the Weibull model is given by

$$I(\theta) = \begin{pmatrix} \beta^2/\alpha^2 & a/\alpha \\ a/\alpha & b^2/\beta^2 \end{pmatrix}$$

where $a = -(1-\gamma)$ and $b^2 = \pi^2/6 + (1-\gamma)^2$ where

$$\gamma = .577216... = \lim_{m\to\infty} \{\sum_{k=1}^{m} \frac{1}{k} - \log m\}$$

is Euler's constant.

5. (40 points) Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta \subset \mathbb{R}^d\}$ is a regular parametric model with densities p_θ with respect to a dominating measure μ. Suppose that $\theta_0 \in \Theta$ is fixed.
   A. Express $H^2(P_\theta, P_{\theta_0})$ in terms of the densities p_{θ_0} and p_θ.
   B. If $g(\theta) = H^2(P_\theta, P_{\theta_0})$, compute

$$\dot{g}(\theta) = \nabla g(\theta) = (\frac{\partial}{\partial\theta_1}g(\theta), \ldots, \frac{\partial}{\partial\theta_d}g(\theta))^T$$

   and the $d \times d$ matrix

$$\ddot{g}(\theta) = \left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}g(\theta)\right),$$

   and $\dot{g}(\theta_0)$ and $\ddot{g}(\theta_0)$ (express the latter in terms of scores and information if possible). You may assume that the necessary interchanges of integration and differentiation are permissible.
   C. Suppose that $\tilde{\theta}_n$ is an estimator satisfying $\sqrt{n}(\tilde{\theta}_n - \theta_0) \rightarrow_d \underline{D} \sim N_d(0, I^{-1}(\theta_0))$ under P_{θ_0}, and define $\tilde{h}_n^2 \equiv 8ng(\tilde{\theta}_n)$. Find the limiting distribution of $\tilde{h}_n^2$ under P_{θ_0}.
   D. Carry out the computations in A, B, and C explicitly when $d = 1$ and $p_\theta(x) = \theta\exp(-\theta x)1_{(0,\infty)}(x)$ with $\theta > 0$.

6. (40 points) A. Compute $H^2(P_1, P_\theta)$, $d_{TV}(P_1, P_\theta)$, $K(P_1, P_\theta)$, and $K(P_\theta, P_1)$ when $P_\theta = \text{Uniform}[0, \theta]$ with $\theta > 0$. [Hint: consider computing $H^2(P, Q)$ and $d_{TV}(P, Q)$ by way of the corresponding affinities

$$\rho(P,Q) = \int \sqrt{pq}\,d\mu, \qquad \text{and} \qquad \eta(P,Q) = \int p \wedge q\,d\mu.]$$

   B. If P_θ^n denotes the probability measure (distribution) corresponding to $X_1, \ldots, X_n$ i.i.d. as $P_\theta = \text{Uniform}[0, \theta]$ show that for $\theta_n = 1 + t/n$ with $t \in \mathbb{R}$ fixed, it follows that $H^2(P_{\theta_n}^n, P_1^n) \rightarrow$ something and find "something".
   C. In Chapter 3, section 5, we established a "basic lower bound inequality" as follows: for any estimator T_n of $\nu(P)$

$$\begin{aligned} &\max\{E_{n,1}l(|T_n - \nu(P_1)|),\ E_{n,2}l(|T_n - \nu(P_2)|)\} \\ &\quad \geq l(\frac{1}{4}|\nu(P_1) - \nu(P_2)|)\{1 - H^2(P_1, P_2)\}^{2n}. \end{aligned}$$

   Apply this basic lower bound in the context of B above with $P_1 = \text{Uniform}[0, 1]$, $P_2 = \text{Uniform}[0, \theta_n]$ with $\theta_n = 1 + t/n$, $l(x) \equiv x$, and $\nu(P_\theta) = \theta$. Conclude that for any estimators T_n of θ for $\theta_0 = 1$ we have

$$\liminf_{n\to\infty} \max\{nE_1|T_n - 1|,\ nE_{\theta_n}|T_n - \theta_n|\} \geq LB(t) > 0$$

   and find $LB(t)$ as a function of t.