STATISTICS 581:
Solutions, Day 1 Quiz, Fall 2005

1. **State** the (classical) Central Limit Theorem.
Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$, then

$$\sqrt{n}(\overline{X}_n - \mu) \to_d N(0, \sigma^2) .$$

In other words, if $Z \sim N(0,1)$ and Φ denotes the distribution function of a standard normal random variable,

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2) dy ,$$

then, for all $t \in R$,

$$P(\sqrt{n}(\overline{X}_n - \mu) \le t) \to P(\sigma Z \le t) = \Phi(t/\sigma) . \qquad (1)$$

[Note: the hypothesis $\sigma^2 < \infty$ is also necessary: if i.i.d. random variables X_i satisfy (1) then $\sigma^2 < \infty$.]

2. **State** the weak law of large numbers.
Solution: If $X_1, \ldots, X_n$ are i.i.d. with $E|X_1| < \infty$ and $E(X_1) = \mu$, then $\overline{X}_n = n^{-1} \sum_1^n X_i \to_p \mu$.

3. Suppose that X has density $f(x) = x^{-2} 1_{[1,\infty)}(x)$.
(a) For what values of $r \in R$ is it true that $E(X^r) < \infty$?
(b) For the values of r for which the expectation is finite, compute it explicitly.
(c) If $X_1, X_2, \ldots$ are i.i.d. with the same density as X, does the law of large numbers hold?
(d) If $X_1, X_2, \ldots$ are i.i.d. with the same density as X, does the central limit theorem hold?
Solution: (a) $E(X^r) = \int_1^\infty x^r x^{-2} dx = \int_1^\infty x^{r-2} dx = x^{r-1}/(r-1)|_1^\infty = 1/(1-r) < \infty$ if $r < 1$. If $r > 1$ then $E(X^r) = \infty$ by the same calculation, while if $r = 1$, then $E(X) = \int_1^\infty x^{-1} dx = \log x|_1^\infty = \infty$.
(b) As shown in (a) $E(X^r) = 1/(1-r)$ for $r < 1$.
(c) No, since $E(X) = \infty$.
(d) No, since $E(X^2) = \infty$.

4. **Define** what is meant by:
A. X_n converges in distribution to X for random variables X and X_n, $n \ge 1$.
B. X_n converges in probability to X.

Solution: A. X_n converges in distribution to X if

$$F_n(x) = P(X_n \le x) \to P(X \le x) = F(x)$$

for all $x \in C_F = \{x \in R : \ F \text{ is continuous at } x\}$.
B. X_n converges in probability to X if

$$P(|X_n - X| > \epsilon) \to 0 \qquad \text{as} \ \ n \to \infty \ \ \text{for every} \ \ \epsilon > 0\,.$$

5. Suppose that U is a random variable with a Uniform$(0,1)$ distribution. For each integer $n \ge 1$ define $X_n = n1_{[0,1/n]}(U)$.
(a) Does $X_n \to_d X$? (If so, identify the distribution of the limiting variable X.)
(b) Does $X_n \to_p X$? (If so, identify the limit variable X.)
(c) Compute $E(X_n)$. Does it converge to $E(X)$?

Solution: Here it is easier to answer (b) first, then (a):
(b) Note that for any $\epsilon \in (0,1)$

$$P(|X_n| > \epsilon) = P(U \le 1/n) = 1/n \to 0\,.$$

Hence $X_n \to_p 0$. (a) Since convergence in probability implies convergence in distribution, the result of (b) implies that $X_n \to_d 0$. [In fact, $X_n \to_{a.s.} 0$: for $U(\omega) \in (0,1]$ we have $1/n < U(\omega)$ for all $n \ge N(\omega)$ sufficiently large, and hence $X_n(\omega) = 0$ for $n \ge N(\omega)$. But $P(U \in (0,1]) = 1$, so it follows that $X_n \to_{a.s.} 0$. Since $\to_{a.s.}$ implies $\to_p$, this also yields the desired conclusion(s).]
(c) The expectation is

$$E(X_n) = E\{n1_{[0,1/n]}(U)\} = nP(0 \le U \le 1/n) = n(1/n) = 1$$

for all n. This gives an example for which $\lim_n E(X_n) \ne E(\lim_n X_n)$. But note that $X_n \ge 0$ and Fatou's lemma (Theorem 0.2.2, page 9, Chapter 0 notes) does indeed hold with strict inequality:

$$0 = E(0) = E(\underline{\lim} X_n) < \underline{\lim} E(X_n) = \underline{\lim} 1 = 1\,.$$

6. Suppose that $X, \ X_1, \ldots, X_n, \ldots$ are independent and identically distributed Exponential(θ) random variables (i.e. $P(X > x) = \exp(-\theta x)$ for $x \ge 0$). Let $T_n = X_1 + \ldots + X_n$.
(a) What is the distribution of T_n?
(b) Compute $E(X)$ and $Var(X)$.
(c) Does $\overline{X}_n = n^{-1} T_n \to_p$ something? If so, what is "something"?
(c) Does $\sqrt{n}(\overline{X}_n - \theta^{-1}) \to_d$ something? If so, what is "something"?
(d) What is the Cramér-Rao bound for unbiased estimators of θ?

Solution: (a) $T_n \sim$ Gamma(n, θ).
(b) For arbitrary $r > 0$,

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r \theta \exp(-\theta x) dx \\ &= \theta^{-r} \int_0^\infty (\theta x)^r \exp(-\theta x) d(\theta x) \\ &= \theta^{-r} \int_0^\infty y^r \exp(-y) dy \qquad \text{by the change of variables } y = \theta x \\ &= \theta^{-r} \Gamma(r+1). \end{aligned}$$

Therefore $E(X) = \theta^{-1}\Gamma(2) = \theta^{-1}1! = \theta^{-1}$ while $E(X^2) = \theta^{-2}\Gamma(3) = \theta^{-2}2$. It follows easily that $Var(X) = \theta^{-2}$.

(c) By the weak law of large numbers, $\overline{X}_n \rightarrow_p \theta^{-1}$.

(d) By the central limit theorem, $\sqrt{n}(\overline{X}_n - \theta^{-1}) \rightarrow_d N(0, 1/\theta^2)$.

(e) The Cramér - Rao bound for unbiased estimators of θ is given by $1/(nI(\theta))$ where

$$I(\theta) = E\{\dot{\mathrm{l}}_\theta^2(X)\} = E\left\{\frac{1}{\theta} - X\right\}^2 = Var(X) = \theta^{-2}.$$

Hence the Cramér - Rao lower bound in this case is θ^2/n. Alternatively,

$$I(\theta) = -E\{\ddot{\mathrm{l}}_{\theta\theta}(X)\} = -E\{-1/\theta^2\} = 1/\theta^2.$$