

Statistics 581, Problem Set 7 Solutions

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1. Compute and plot the *score for location*, $-(f'/f)(x)$ when:
- A. $f(x) = \phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, (normal or Gaussian);
 - B. $f(x) = \exp(-x)/(1 + \exp(-x))^2$, (logistic);
 - C. $f(x) = \frac{1}{2} \exp(-|x|)$, (double exponential);
 - D. $f = t_k$, the t -distribution with k degrees of freedom;
 - E. $f(x) = \exp(-x) \exp(-\exp(-x))$, Gumbel or extreme value.

Soluton: A. For $f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, it follows that $\log f(x) = -x^2/2 + \text{constant}$ so that $(-f'/f)(x) = x$, $-1 - x(f'/f)(x) = x^2 - 1$.

B. For $f(x) = e^{-x}/(1 + e^{-x})^2$, $\log f(x) = -x - 2 \log(1 + e^{-x})$ and

$$-\frac{f'}{f}(x) = \frac{1 - e^{-x}}{1 + e^{-x}},$$

while

$$-1 - x \frac{f'}{f}(x) = x \frac{1 - e^{-x}}{1 + e^{-x}} - 1 \sim |x| - 1 \quad \text{as} \quad |x| \rightarrow \infty.$$

C. For $f(x) = 2^{-1} \exp(-|x|)$,

$$\log f(x) = -|x| + \text{constant},$$

and

$$-\frac{f'}{f}(x) = \begin{cases} -1 & x < 0 \\ \text{undefined} & x = 0 \\ +1 & x > 0 \end{cases},$$

while

$$-1 - x \frac{f'}{f}(x) = |x| - 1, \quad \text{for} \quad x \neq 0.$$

D. For the t_k distribution, $f(x) = \frac{\Gamma(\frac{1}{2}(k+1))}{\Gamma(\frac{1}{2}k)} \frac{1}{\sqrt{\pi k}} (1 + \frac{x^2}{k})^{-(k+1)/2}$,

$$\log f(x) = -\frac{k+1}{2} \log(1 + \frac{x^2}{k}),$$

and

$$-\frac{f'}{f}(x) = \frac{k+1}{k} \frac{x}{1 + \frac{x^2}{k}},$$

while

$$-1 - x \frac{f'}{f}(x) = k \frac{x^2 - 1}{x^2 + k}.$$

E. For $f(x) = \exp(-x) \exp(-\exp(-x))$,

$$\log f(x) = -x - \exp(-x),$$

and

$$-\frac{f'}{f}(x) = 1 - \exp(-x),$$

while

$$-1 - x\frac{f'}{f}(x) = -1 + x(1 - \exp(-x)).$$

Plots of these score functions for location are given in Figure 1. Note that they are *odd functions* in cases A-D, which are all symmetric densities about zero. In case E, corresponding to the asymmetric extreme value density, the score for location does not have any obvious symmetry property.

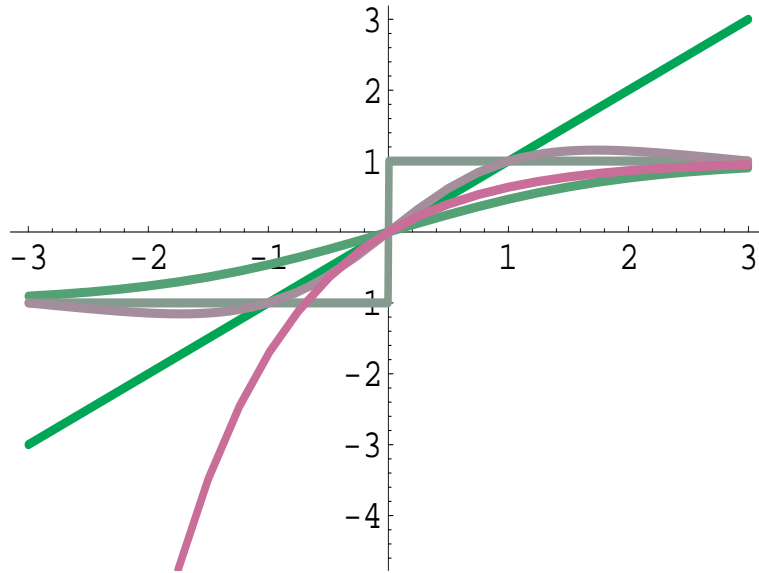


Figure 1: Scores for location.

2. Compute $I_f = \int (f'(x)/f(x))^2 f(x) dx$, the information for location, for each of the densities in problem 1.

Solution: A. In this case $I_f = \int x^2 \phi(x) dx = \text{Var}(Z) = 1$ where $Z \sim N(0, 1)$.

B. For the logistic density the information for location is

$$\begin{aligned} I_f &= \int_{-\infty}^{\infty} \left(\frac{1 - e^{-x}}{1 + e^{-x}} \right)^2 dF(x) = \int_{-\infty}^{\infty} (2F(x) - 1)^2 dF(x) \\ &= \int_0^1 (2u - 1)^2 du = 4\text{Var}(U) = 4 \frac{1}{12} = \frac{1}{3}. \end{aligned}$$

C. For the double-exponential density, $[(-f'/f)(x)]^2 = 1$, so $I_f = 1$.

D. For the t - distribution with k degrees of freedom, by using a change of variables and letting T_r denote a random variable with the t - distribution with

r degrees of freedom,

$$\begin{aligned}
 I_f &= \int_{-\infty}^{\infty} \left(\frac{k+1}{k}\right)^2 \frac{x^2}{(1+x^2/k)^2} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{k}{2})\sqrt{\pi k}} \frac{1}{(1+x^2/k)^{(k+1)/2}} dx \\
 &= \frac{(k+1)(k+2)}{(k+4)(k+3)} \text{Var}(T_{k+4}) \\
 &= \frac{(k+1)(k+2)k+4}{(k+4)(k+3)k+2} = \frac{k+1}{k+3}
 \end{aligned}$$

since $\text{Var}(T_r) = r/(r-2)$ for $r > 2$.

E. For the extreme value distribution $F(x) = \exp(-\exp(-x))$ and therefore if $X \sim F$, the random variable $Y \equiv \exp(-X) \sim \text{exponential}(1)$:

$$\begin{aligned}
 P(Y \geq y) &= P(\exp(-X) \geq y) = P(X \leq -\log(y)) \\
 &= \exp(-\exp(\log(y))) = \exp(-y).
 \end{aligned}$$

Since $-(f'/f)(x) = -1 + e^{-x}$, it is easy to see that

$$I_f = E\left[-\frac{f'}{f}(X)\right]^2 = E[\exp(-X) - 1]^2 = E[Y - 1]^2 = \text{Var}(Y) = 1.$$

3. Consider the two parameter location-scale model

$$\mathcal{P} = \{P_\theta : \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta\}$$

where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right),$$

and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ .

(a) Calculate the information matrix $I(\theta)$ for θ .

(b) For which of the densities in A-E of problem 1 is $I_{12}(\theta)$ not zero?

Solution: (a) The score for location is

$$\begin{aligned}
 \dot{\mathbf{i}}_1(x) &= \frac{\partial}{\partial \theta_1} \log \left\{ \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) \right\} \\
 &= -\frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \frac{1}{\theta_2}
 \end{aligned}$$

and the score for scale is

$$\begin{aligned}
 \dot{\mathbf{i}}_2(x) &= \frac{\partial}{\partial \theta_2} \log \left\{ \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right) \right\} \\
 &= -\frac{1}{\theta_2} - \frac{f'}{f}\left(\frac{x - \theta_1}{\theta_2}\right) \frac{(x - \theta_1)}{\theta_2^2}
 \end{aligned}$$

Thus we compute

$$\begin{aligned}
I_{11}(\theta) &= E\dot{\mathbf{i}}_1^2(X) = \frac{1}{\theta_2^2} \int \left(\frac{f'}{f} \left(\frac{x - \theta_1}{\theta_2} \right) \right)^2 \frac{1}{\theta_2} f \left(\frac{x - \theta_1}{\theta_2} \right) dx \\
&= \frac{1}{\theta_2^2} \int \left(\frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{f,loc}, \\
I_{22}(\theta) &= E\dot{\mathbf{i}}_2^2(X) = \frac{1}{\theta_2^2} \int \left(-1 - \frac{(x - \theta_1)}{\theta_2} \frac{f'}{f} \left(\frac{x - \theta_1}{\theta_2} \right) \right)^2 \frac{1}{\theta_2} f \left(\frac{x - \theta_1}{\theta_2} \right) dx \\
&= \frac{1}{\theta_2^2} \int \left(-1 - y \frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{f,scal}, \\
I_{12}(\theta) &= E\dot{\mathbf{i}}_1(X)\dot{\mathbf{i}}_2(X) = I_{21}(\theta) \\
&= \frac{1}{\theta_2^2} \int \left(-\frac{f'}{f} \left(\frac{x - \theta_1}{\theta_2} \right) \right) \left(-1 - \frac{(x - \theta_1)}{\theta_2} \frac{f'}{f} \left(\frac{x - \theta_1}{\theta_2} \right) \right) \frac{1}{\theta_2} f \left(\frac{x - \theta_1}{\theta_2} \right) dx \\
&= \frac{1}{\theta_2^2} \int y \left(\frac{f'}{f}(y) \right)^2 f(y) dy \equiv \frac{1}{\theta_2^2} I_{12,f}.
\end{aligned}$$

(b) Note that $I_{12,f} = 0$ for all symmetric densities f since y is odd while $(f'/f)^2 f$ is even. Thus $I_{12}(\theta) = 0$ for cases A-D in problem 1, while $I_{12}(\theta) = \theta_2^{-2} I_{12,f} \neq 0$ in case E. I calculate

$$\begin{aligned}
I_{12,f} &= \int y \left(\frac{f'}{f}(y) \right)^2 f(y) dy \\
&= \int_{-\infty}^{\infty} y(-1 + e^{-y})^2 e^{-y} \exp(-e^{-y}) dy \\
&= - \int_0^{\infty} (\log v)(v - 1)^2 e^{-v} dv = -(1 - \gamma)
\end{aligned}$$

where γ is Euler's constant. In fact this is related to I_{12} that we calculated already for the Weibull family in Example xx.xx in the notes and in the first display at the top of page 2 of the Handout on Gamma, Digama, and Polygamma.

4. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \leq m < k$. Let $\dot{\mathbf{i}} = \dot{\mathbf{i}}_\theta = (\dot{\mathbf{i}}_1, \dot{\mathbf{i}}_2)$ be the corresponding partition of the (vector of) scores $\dot{\mathbf{i}}$, and, with $\tilde{\mathbf{i}} \equiv I^{-1}(\theta)\dot{\mathbf{i}}$, the *efficient influence function* for θ , let $\tilde{\mathbf{i}} = (\tilde{\mathbf{i}}_1, \tilde{\mathbf{i}}_2)$ be the corresponding partition of $\tilde{\mathbf{i}}$. In both cases, $\dot{\mathbf{i}}_1, \tilde{\mathbf{i}}_1$ are m -vectors of functions, and $\dot{\mathbf{i}}_2, \tilde{\mathbf{i}}_2$ are $k - m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where I_{11} is $m \times m$, I_{12} is $m \times (k - m)$, I_{21} is $(k - m) \times m$, I_{22} is $(k - m) \times (k - m)$. Also write

$$I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}.$$

Verify that:

A. $I^{11} = I_{11.2}^{-1}$ where $I_{11.2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21}$,
 $I^{22} = I_{22.1}^{-1}$ where $I_{22.1} \equiv I_{22} - I_{21}I_{11}^{-1}I_{12}$,
 $I^{12} = -I_{11.2}^{-1}I_{12}I_{22}^{-1}$,
 $I^{21} = -I_{22.1}^{-1}I_{21}I_{11}^{-1}$.

This amounts to formulas (5) and (6) of section 3.2, page 15.

B. Verify that

$$\tilde{\mathbf{i}}_1 = I^{11}\mathbf{i}_1 + I^{12}\mathbf{i}_2 = I_{11.2}^{-1}(\mathbf{i}_1 - I_{12}I_{22}^{-1}\mathbf{i}_2), \text{ and}$$

$$\tilde{\mathbf{i}}_2 = I^{21}\mathbf{i}_1 + I^{22}\mathbf{i}_2 = I_{22.1}^{-1}(\mathbf{i}_2 - I_{21}I_{11}^{-1}\mathbf{i}_1).$$

The first of these is (7) on page 15, section 3.2.

Solution: A. This is just block inversion/multiplication of matrices:

$$\begin{aligned} \begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} &= \begin{pmatrix} I_{11.2}^{-1} & -I_{11.2}^{-1}I_{12}I_{22}^{-1} \\ -I_{22.1}^{-1}I_{21}I_{11}^{-1} & I_{22.1}^{-1} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_{11.2}^{-1}(I_{11} - I_{12}I_{22}^{-1}I_{21}) & I_{11.2}^{-1}(I_{12} - I_{12}I_{22}^{-1}I_{22}) \\ I_{22.1}^{-1}(-I_{21} + I_{21}) & I_{22.1}^{-1}(-I_{21}I_{11}^{-1}I_{12} + I_{22}) \end{pmatrix} \\ &= \begin{pmatrix} Ident & 0 \\ 0 & Ident \end{pmatrix} = Identity. \end{aligned}$$

by using the definition of $I_{11.2}$ and $I_{22.1}$.

B. This follows immediately from the formulas for I^{11} and I^{12} by just plugging into the formula $\tilde{\mathbf{i}}_1 = I^{11}\mathbf{i}_1 + I^{12}\mathbf{i}_2$ for $\tilde{\mathbf{i}}_1$:

$$\begin{aligned} \tilde{\mathbf{i}}_1 &= I_{11.2}^{-1}\mathbf{i}_1 - I_{11.2}^{-1}I_{12}I_{22}^{-1}\mathbf{i}_2 \\ &= I_{11.2}^{-1}(\mathbf{i}_1 - I_{12}I_{22}^{-1}\mathbf{i}_2) = I_{11.2}^{-1}\mathbf{i}_1^*. \end{aligned}$$