Statistics 581, Problem Set 2 Solutions

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1. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}\,;$$

i.e.

$$E(Y-EY)^2 = E\{E[(Y-E(Y|X))^2|X]\} + E\{[E(Y|X)-E(Y)]^2\}\,.$$

(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi^2_n(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

Solution: (a) We compute directly:

$$\begin{aligned} Var(Y) &= E[Y-E(Y)]^2 = E[Y-E(Y|X)+E(Y|X)-E(Y)]^2 \\ &= E[Y-E(Y|X)]^2 + 2E[(Y-E(Y|X)][E(Y|X)-E(Y)] \\ &\qquad + E[E(Y|X)-E(Y)]^2 \\ &= E\{E\{[Y-E(Y|X)]^2|X\}\} + 0 + Var[E(Y|X)] \\ &= E\{Var[Y|X]\} + Var[E(Y|X)] \end{aligned}$$

since, by computing conditionally,

$$\begin{aligned} &E[(Y-E(Y|X)][E(Y|X)-E(Y)] \\ &\quad = E\{E\{[(Y-E(Y|X)][E(Y|X)-E(Y)]|X\}\} \\ &\quad = E\{[E(Y|X)-E(Y)]E\{[Y-E(Y|X)]|X\}\} \\ &\quad = E\{[E(Y|X)-E(Y)]\{E(Y|X)-E(Y|X)\}\} \\ &\quad = E\{[E(Y|X)-E(Y)]\cdot 0\} \\ &\quad = 0\,. \end{aligned}$$

(b) A geometric interpretation of (a) is that $Y-E(Y|X)$ is orthogonal to $E(Y|X)-E(Y)$ in $L_2(\Omega,\mathcal{A},P) = L_2(P)$, thus the identity in (a) can be intepreted as a "pythagorean theorem". Also note that $Y - E(Y|X)$ is orthogonal to any function $g(X)$: much as in the last part of (a)

$$\begin{aligned} & E[(Y - E(Y|X)]g(X) \\ & \quad = E\{E\{[(Y - E(Y|X)]g(X)|X\}\} \\ & \quad = E\{g(X)E\{[Y - E(Y|X)]|X\}\} \\ & \quad = E\{g(X)\{E(Y|X) - E(Y|X)\}\} \\ & \quad = E\{g(X) \cdot 0\} \\ & \quad = 0\,. \end{aligned}$$

(c) Now $(Y|K) \sim \chi^2_{2K+n}$ where $K \sim \text{Poisson}(\delta/2)$, so

$$E(Y) = E\{E(Y|K)\} = E\{2K + n\} = n + 2(\delta/2) = n + \delta\,.$$

Furthermore, using part (a) we get

$$\begin{aligned} Var(Y) &= E\{Var(Y|K)\} + Var\{E(Y|K)\} \\ &= E\{2(2K+n)\} + Var\{2K+n\} \\ &= 4(\delta/2) + 2n + 4(\delta/2) \\ &= 2n + 4\delta\,. \end{aligned}$$

2. Ferguson, ACILST, #1, page 11.
Let $X_1, X_2, \ldots$ be i.i.d. random variables with densities $f(x) = \alpha x^{-(\alpha+1)} 1_{(1,\infty)}(x)$.
(a) For what values of $\alpha > 0$ and $r > 0$ is it true that $n^{-1}X_n \to_r 0$?
(b) For what values of $\alpha > 0$ is it true that $n^{-1}X_n \to_{a.s.} 0$?
(c) If $X_1, X_2, \ldots$ are independent with X_n having density $f_n(x) = \alpha_n x^{-(\alpha_n+1)} 1_{(1,\infty)}(x)$ for $n = 1, 2, \ldots$, Find the limit of $n^{-2}EX_n^2$ when $\alpha_n = 2 + n^{-\gamma}$ for $\gamma \in \mathbb{R}$.

Solution: (a) $E(X_n^r) = \alpha \int_1^\infty x^r x^{-(\alpha+1)} dx = \alpha/(\alpha - r)$ if $\alpha > r$, while $E(X_n^r) = \infty$ if $\alpha \le r$. Thus $E[(n^{-1}X_n)^r] = n^{-r}(\alpha/(\alpha - r)) \to 0$ if $\alpha > r$.
(b) $P(n^{-1}X_n > \epsilon) = P(X_n > n\epsilon) = \alpha \int_{n\epsilon}^\infty x^{-(\alpha+1)} dx = (n\epsilon)^{-\alpha}$, so for $\alpha > 1$ we have

$$\sum_{n=1}^\infty P(n^{-1}X_n > \epsilon) \le \sum_{n=1}^\infty (n\epsilon)^{-\alpha} < \infty.$$

Thus $P(n^{-1}X_n > \epsilon \text{ i.o.}) = 0$ by the Borel-Cantelli lemma, and we conclude that $n^{-1}X_n \rightarrow_{a.s.} 0$ for $\alpha > 1$. [Since the X_n's are independent, $n^{-1}X_n \rightarrow_{a.s.} 0$ if *and only if* $\alpha > 1$ by the converse Borel-Cantelli lemma.]

(c) If the X_n's are independent with the same densities as in (a) and (b) but with $\alpha = \alpha_n = 2 + n^{-\gamma}$ for X_n, then $EX_n^2 = (2 + n^{-\gamma})/n^{-\gamma}$, so

$$n^{-2}EX_n^2 = (2 + n^{-\gamma})/n^{2-\gamma} \rightarrow \begin{cases} 0, & \text{if } \gamma < 2, \\ 2, & \text{if } \gamma = 2, \\ \infty, & \text{if } \gamma > 2. \end{cases} .$$

3. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi_k^2(\delta)$ with $\delta = \mu'\mu$.

Solution: See Ferguson, ACILST, page 63 (and page 57). Find Γ orthogonal so that $\Gamma'\Sigma\Gamma = D$ where D is diagonal. Now $\Gamma Gamma' = I$, so if $\Sigma^2 = \Sigma$ we have $D^2 = \Gamma'\Sigma\Gamma\Gamma'\Sigma\Gamma = \Gamma'\Sigma^2\Gamma = \Gamma'\Sigma\Gamma = D$ and conversely. Moreover, since Σ is of rank r, D is of rank r, and this together with $D^2 = D$ implies that D has r 1's on the diagonal and $n - r$ 0's. Without loss, assume that Γ has been chosen so that the r ones occur in the the first r positions of the diagonal matrix D; thus

$$D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is $r \times r$. Moreover, note that

$$\begin{aligned} D\Gamma'\mu &= \Gamma'\Sigma\Gamma\Gamma'\mu = \Gamma'\Sigma\mu \\ &= \Gamma\mu, \end{aligned}$$

and this implies that the last $n - r$ components of $\Gamma'\mu$ are all zero. Now let $Y = \Gamma'X$ (much as in the proof of theorem 1.3.2 of the notes). Then $Y \sim N_n(\Gamma'\mu, D)$, $Y'Y = X'\Gamma\Gamma'X = X'X$, and by (d) of page 16, section 1.3, $X'X = Y'Y \sim \chi_r^2(\delta)$ where $\delta = (\Gamma'\mu)'(\Gamma'\mu) = \mu'\Gamma\Gamma'\mu$.

4. Ferguson, ACILST, #1, page 6:
(a) If $X_n \sim \text{Beta}(1/n, 1/n)$ and $X \sim \text{Bernoulli}(1/2)$, show that $X_n \to_d X$.
(b) Suppose that $X_n \sim \text{Beta}(\alpha/n, \beta/n)$ with $\alpha, \beta > 0$. Does $X_n \to_d X$ for some X?

Solution: (a) When $X_n \sim \text{Beta}(1/n, 1/n)$, the density of X_n is $f_n(x) = c_n x^{1/n-1}(1-x)^{1/n-1}$ where $c_n = \Gamma(2/n)/\Gamma(1/n)^2$. Since $\Gamma(1+y) = y\Gamma(y)$ for all $y > 0$, and $\Gamma(1) = 0! = 1$,

$$f_n(x) = \frac{\Gamma(1+2/n)/(2/n)}{[\Gamma(1+1/n)/(1/n)]^2} x^{1/n-1}(1-x)^{1/n-1} \sim \frac{1}{2n} x^{-1}(1-x)^{-1} \to 0$$

for all $0 < x < 1$ (where $\sim$ means that the ratio of the two sides converges to 1 for all fixed $x \in (0,1)$ as $n \to \infty$). Thus $P(\epsilon \le X_n \le 1-\epsilon) \to 0$ as $n \to \infty$. By symmetry $p_n \equiv P(X_n \le \epsilon) = P(X_n \ge 1-\epsilon)$ and hence $2p_n \to 1$ or $p_n \to 1/2$. Thus it follows that the d.f. F_n of X_n satisfies $F_n(x) \to 1/2 = F(x)$ for all $0 < x < 1$ where F is the d.f. of a Bernoulli(1/2) random variable. Since the convergence holds trivially for $x \in [0,1]^c$, it follows that $X_n \to_d X$.
(b) When $X_n \sim \text{Beta}(\alpha/n, \beta/n)$, then the same sort of computation as in (a) yields

$$\begin{aligned} f_n(x) &= \frac{\Gamma(1+(\alpha+\beta)/n)/((\alpha+\beta)/n)}{\Gamma(1+\alpha/n)/(\alpha/n)\Gamma(1+\beta/n)/(\beta/n)} x^{\alpha/n-1}(1-x)^{\beta/n-1}, \quad 0 < x < 1 \\ &\sim \frac{\alpha\beta}{n(\alpha+\beta)} x^{-1}(1-x)^{-1} \to 0, \end{aligned}$$

so $P(\epsilon < X_n < 1-\epsilon) \to 0$ for every $0 < \epsilon \le 1/2$ as before. But

$$\begin{aligned} P(X_n \le x) &= \int_0^x f_n(y)dy \\ &\sim \frac{\alpha\beta}{n(\alpha+\beta)} \int_0^x y^{\alpha/n-1}(1-y)^{\beta/n-1}dy \\ &\quad \begin{cases} \le \frac{\alpha\beta}{n(\alpha+\beta)} \int_0^x y^{\alpha/n-1}dy(1-x)^{\beta/n-1} = \frac{\beta}{\alpha+\beta} x^{\alpha/n}(1-x)^{\beta/n-1} \\ \ge \frac{\alpha\beta}{n(\alpha+\beta)} \int_0^x y^{\alpha/n-1}dy = \frac{\beta}{\alpha+\beta} x^{\alpha/n} \end{cases} \\ &\to \begin{cases} \frac{\beta}{\alpha+\beta}(1-x)^{-1} \\ \frac{\beta}{\alpha+\beta} \end{cases} . \end{aligned}$$

But for x arbitrarily close to zero the upper and lower bounds become equal (in the limit) and the common value is $\beta/(\alpha+\beta)$. Combined with $P(\epsilon < X_n < 1-\epsilon) \to 0$, this implies that $P(X_n \le x) \to \beta/(\alpha+\beta)$ for $0 < x < 1$, and hence $X_n \to_d \text{Bernoulli}(\alpha/(\alpha+\beta))$. [Note the slight difference with Ferguson's solution at this point; see ACILST page 172, line -7.]

5. (a) Ferguson, ACILST, #4, page 6:
Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.
(b) Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to \infty$.
(c) Give an example of a sequence of random variables X_n for which $X_n \to_p 0$ but $X_n \to_{a.s.} 0$ fails.

Solution: (a) If $X_n = a_n$ with probability p_n and $X_n = 0$ with probability $1-p_n$, then $E(X_n) = a_n p_n$ and $E(X_n^2) = a_n^2 p_n = 1$ if $p_n = 1/a_n^2$. Then $E(X_n) = a_n/a_n^2 = 1/a_n \to 0$ if $a_n \to \infty$. Ferguson's solution on page 173 takes $a_n = n$; the same holds for any sequence $a_n \to \infty$.
(b) Let $U \sim \text{Uniform}(0,1)$, and set $X_n = n^\alpha 1_{[0,1/n]}(U)$. Then $EX_n = n^\alpha n^{-1} \to 0$ if $\alpha < 1$, while $EX_n^2 = n^{2\alpha} n^{-1} \to \infty$ if $\alpha > 1/2$. Thus the required convergences hold for all $1/2 < \alpha < 1$.
(c) Let $U \sim \text{Uniform}(0,1)$. The "dancing functions" are defined by $X_{n,k} = 1_{[(k-1)/2^n, k/2^n)}(U)$, $k = 1, \ldots, 2^n$, $n = 1, 2, \ldots$. Let $\{Y_m\}_{m \ge 1}$ be defined by $Y_m = X_{n,k}$ if $m = (\sum_{j=1}^n 2^j) + k = 2^{n+1} - 2 + k$ with $1 \le k \le 2^n$. Then for $\epsilon \in (0,1)$,

$$P(|Y_m| > \epsilon) = P(|X_{n,k}| > \epsilon) = 2^{-n} \to 0$$

so $Y_m \to_p 0$, but for every $U(\omega) \in (0,1)$ we have $Y_m(\omega) = 1$ for infinitely many m's and also $Y_m(\omega) = 0$ for infinitely many m's. Hence

$$0 = \liminf Y_m < \limsup Y_m = 1 \qquad a.s.$$

and it follows that Y_m does not converge to 0 almost surely.

6. Suppose that $U \sim \text{Uniform}(0,1)$, $\alpha > 0$, and

$$X_n \equiv (n^\alpha / \log(n+1)) 1_{[0,1/n^\alpha]}(U)\,.$$

(a) Show that $X_n \to_{a.s.} 0$ and $E(X_n) \to E(0) = 0$.

(b) Can you find a random variable Y with $|X_n| \le Y$ for all n with $E(Y) < \infty$ for any α?
(c) For what values of α does the uniform integrability condition

$$\limsup_{n\to\infty} E\{|X_n| 1_{[|X_n|\ge M]}\} \to 0 \qquad \text{as} \;\; M \to \infty$$

hold?

Solution: (a) $X_n \to_{a.s.} 0$ since $X_n(\omega) = 0$ for $1/n^\alpha < U(\omega)$, or equivalently $n > (1/U(\omega))^{1/\alpha}$ and since $P(0 < U \le 1) = 1$. Moreover,

$$E(X_n) = \frac{n^\alpha}{\log(n+1)} \frac{1}{n^\alpha} = \frac{1}{\log(n+1)} \to 0 = E(0)\,.$$

(b) Now the smallest possible random variable Y satisfying $|X_n| \le Y$ for all n is Y defined by

$$Y = \sum_{k=1}^\infty \frac{k^\alpha}{\log(k+1)} 1_{(1/(k+1)^\alpha, 1/k^\alpha]}(U)\,.$$

But we compute

$$\begin{aligned}
E(Y) &= \sum_{k=1}^\infty \frac{k^\alpha}{\log(k+1)} \left\{ \frac{1}{k^\alpha} - \frac{1}{(k+1)^\alpha} \right\} \\
&= \sum_{k=1}^\infty \frac{1}{\log(k+1)} \left\{ 1 - \left(\frac{k}{k+1} \right)^\alpha \right\} \\
&= \sum_{k=1}^\infty \frac{1}{\log(k+1)} \left\{ 1 - \left(1 - \frac{1}{k+1} \right)^\alpha \right\} \\
&\ge \sum_{k=1}^{k(\alpha)} \frac{1}{\log(k+1)} \left\{ 1 - \left(1 - \frac{1}{k+1} \right)^\alpha \right\} \\
&\qquad + \sum_{k=k(\alpha)}^\infty \frac{1}{\log(k+1)} \frac{\alpha/2}{k+1} \\
&= \infty\,.
\end{aligned}$$

since $(1-x)^\alpha \le 1 - \alpha x/2$ for $x \le x(\alpha)$. Thus there is no integrable dominating function Y for any value of α.
(c) On the other hand the uniform integrability condition does hold for any $\alpha > 0$:

$$\begin{aligned}
E\{|X_n| 1_{[|X_n| \ge M]}\} &= E\left\{ \frac{n^\alpha}{\log(n+1)} 1_{[0,1/n^\alpha]}(U) 1\{(n^\alpha / \log(n+1)) \ge M, U \le 1/n^\alpha\} \right\} \\
&= \frac{n^\alpha}{\log(n+1)} E\left\{ 1_{[0,1/n^\alpha]}(U) \right\} 1\{(n^\alpha / \log(n+1)) \ge M\} \\
&= \frac{1}{\log(n+1)} 1\{(n^\alpha / \log(n+1)) \ge M\} \\
&\to 0 \cdot 1 = 0
\end{aligned}$$

as $n \to \infty$ for every $\alpha > 0$ and $M > 0$. Hence the sequence $\{X_n\}$ is uniformly integrable for every $\alpha > 0$. [The point here is that uniform integrability can still hold even though the conditions of the dominated convergence theorem fail.]