Statistics 581, Problem Set 9

Wellner; 11/23/2005

Reading: Chapter 4, Sections 1-2;
Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;
Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.
Due: Wednesday, November 30, 2005.

1. (a) Lehmann and Casella, Problem 2.13, page 501.
   (b) Let $R_n(\theta) \equiv nE_\theta(T_n - \theta)^2$ where T_n is the Hodges superefficient estimator as in Example 3.3.1 (so $T_n = \delta_n$ of Example 2.5, Lehmann and Casella pages 440 - 443). Show that $R_n(n^{-1/4}) \to \infty$ as $n \to \infty$.

2. (a) Lehmann and Casella, Problem 6.6, page 142.
   (b) Consider the somewhat more general family

   $$\mathcal{P} = \{(1-\epsilon)\phi(x-\xi) + (\epsilon/\tau)\phi((x-\mu)/\tau)) : \ \theta = (\epsilon, \xi, \mu, \tau) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+\}$$

   where ϕ is the standard normal density. Is the information matrix for $(\epsilon, \xi, \mu, \tau)$ in this family always nonsingular? [Hint: what is the relationship of the model $\mathcal{P}$ in (b) to the model in Lehmann and Casella's problem 5.21, page 139?]
   (c) Suppose that $(X, \Delta) \in \mathbb{R} \times \{0, 1\}$ has distribution given by $p_\theta(x|\Delta = 0) = \phi(x)$, $p_\theta(x|\Delta = 1) = \phi((x-\mu)/\tau)/\tau$, and $\Delta \sim$ Bernoulli(ϵ). Show that the marginal distributions of X are in the family $\mathcal{P}$ of part (a). What is the relationship of the information for ϵ based on observation of (X, δ) to the information for ϵ based on observation of X alone?

3. (a) Exercise 2.1.6, page 10, chapter 2 notes.
   (b) Exercise 2.1.7, page 10, chapter 2 notes.

4. (a) Lehmann and Casella, problem 6.3.1, page 501.
   (b) Lehmann and Casella, problem 6.3.2, page 501.
   (c) Lehmann and Casella, problem 6.3.4, page 501.

5. Consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities

   $$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}\exp(-(\frac{x}{\alpha})^\beta)1_{[0,\infty)}(x)$$

   where $\theta \equiv (\alpha, \beta) \in (0, \infty) \times (0, \infty) \subset R^2$. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with density function p_θ.
   A. If $X \sim P_\theta \in \mathcal{P}$, show that the distributions of $\log X$ form a location and scale family from a Gumbel (extreme value) density on R.
   B. Use the result of A to construct method of moments estimators or quantile based estimators $\overline{\theta}_n$ of $\theta = (\alpha, \beta)$.

C. Show that the method of moments or quantile estimators $\overline{\theta}_n$ of θ are asymptotically normal, and find the asymptotic distribution; i.e. show that

$$\sqrt{n}(\overline{\theta}_n - \theta) \to_d N_2(0, \Sigma) \qquad \text{for some} \qquad \Sigma.$$

[We will use these estimators as "starting points" approximate (or one-step) maximum likelihood estimators in problem set 10.]

6. **Optional bonus problem:** Lehmann and Casella, TPE, problem 6.3.22, page 503, reworded as follows. (In other words, prove (vi) of theorem 1.5, page 5, chapter 4 notes). Suppose that $X_1, \ldots, X_n$ are i.i.d. with density p_θ, $\theta \in \Theta \subset R^k$, satisfying the hypotheses of theorem 4.1, page 429 (the Cramér conditions given in (A) - (D) on page 429). Show that the following Local Asymptotic Normality (LAN) result holds for the (local) log- likelihood ratios: with

$$L_n(\theta) \equiv \log(\prod_{i=1}^n p_\theta(X_i)) = \sum_{i=1}^n \log p_\theta(X_i),$$

for a fixed $\theta_0 \in \Theta$,

$$\begin{aligned} L_n(\theta_0 + n^{-1/2}\underline{t}) - L_n(\theta_0) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \underline{t}^T \dot{\underline{l}}_\theta(X_i) - \frac{1}{2}\underline{t}^T I(\theta_0)\underline{t} + o_p(1) \\ &\to_d N(0, t^T I(\theta_0)t) - \frac{1}{2} t^T I(\theta_0) t =_d N(-\frac{1}{2}\sigma^2 \sigma^2) \end{aligned}$$

under P_{θ_0} where $\sigma^2 \equiv \underline{t}^T I(\theta_0)\underline{t}$.