Statistics 581, Problem Set 8

Wellner; 11/16/2005

Reading: Chapter 3, Section 2; Section 3, pages 24-25; Sections 4 - 5, pages 35-43
Ferguson, ACLST, Chapter 19, pages 126-132, Chapter 20, pages 133-134;
Lehmann and Casella, pages 113-129, and 439- 443;
Due: Wednesday, November 23, 2002.

1. Suppose that $\theta = (\theta_1, \theta_2) \in \Theta \subset R^k$ where $\theta_1 \in R$ and $\theta_2 \in R^{k-1}$. Show that:
A. $\mathbf{l}_1^* = \dot{\mathbf{l}}_1 - I_{12}I_{22}^{-1}\dot{\mathbf{l}}_2$ is orthogonal to $[\dot{\mathbf{l}}_2] \equiv \{a'\dot{\mathbf{l}}_2 : a \in R^{k-1}\}$ in $L_2(P_\theta)$.
B. $I_{11\cdot 2} = \inf_{c \in R^{k-1}} E_\theta(\dot{\mathbf{l}}_1 - c'\dot{\mathbf{l}}_2)^2$ and that the minimum is achieved when $c' = I_{12}I_{22}^{-1}$. Thus
$$I_{11\cdot 2} = E_\theta(\dot{\mathbf{l}}_1 - I_{12}I_{22}^{-1}\dot{\mathbf{l}}_2)^2 = E_\theta[(\mathbf{l}_\theta^*)^2]\,.$$
C. Prove the formulas (16) and (17) on page 21 of the Chapter 3 notes and interpret these formulas geometrically.

2. Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by
$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp(-\beta x)1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0,\infty) \times (0,\infty) \equiv \Theta\,.$$
Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0; here $F_\theta(x) \equiv P_\theta(X \le x)$.
(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page 127, Table 6.1
(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
(iii) Find the efficient influence functions for estimation of q_A and q_B.
(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators $\overline{X}_n$ and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.

3. Suppose that $(Y|Z) \sim \text{Weibull}(\lambda^{-1}e^{-\gamma Z}, \beta)$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ. Thus the conditional cumulative hazard function $\Lambda(t|z)$ is given by
$$\Lambda_{\gamma,\lambda,\beta}(t|z) = (\lambda e^{\gamma Z} t)^\beta = \lambda^\beta e^{\beta\gamma Z} t^\beta$$
and hence
$$\lambda_{\gamma,\lambda,\beta}(t|z) = \lambda^\beta e^{\beta\gamma Z}\beta t^{\beta-1}\,.$$
(Recall that $\lambda(t) = f(t)/(1-F(t))$ and
$$\Lambda(t) \equiv \int_0^t \lambda(s)ds = \int_0^t (1-F(s))^{-1}dF(s) = -\log(1-F(t))$$
if F is continuous.) Thus it makes sense to reparametrize by defining $\theta_1 \equiv \beta\gamma$ (this is the parameter of interest since it reflects the effect of the covariate Z), $\theta_2 \equiv \lambda^\beta$, and $\theta_3 \equiv \beta$. This yields
$$\lambda_\theta(t|z) = \theta_3\theta_2\exp(\theta_1 z)t^{\theta_3 - 1}$$

You may assume that

$$a(z) \equiv (\partial/\partial\eta) \log g_\eta(z)$$

exists and $E\{a^2(Z)\} < \infty$. Thus Z is a "covariate" or "predictor variable", θ_1 is a "regression parameter" which affects the intensity of the (conditionally) Exponential variable Y, and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ where $\theta_4 \equiv \eta$.

(a) Derive the joint density $p_\theta(y, z)$ of (Y, Z) for the re-parametrized model.

(b) Find the information matrix for θ. What does the structure of this matrix say about the effect of $\eta = \theta_4$ being known or unknown about the estimation of $\theta_1, \theta_2, \theta_3$?

(c) Find the information and information bound for θ_1 if the parameters θ_2 and θ_3 are known?

(d) What is the information bound for θ_1 if just θ_3 is known to be equal to 1?

(e) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_3 is known.

(f) Find the information $I_{11\cdot(2,3)}$ and information bound for θ_1 if the parameters θ_2 and θ_3 are unknown. (Here both θ_2 and θ_3 are in "the second block".)

(g) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_2 and θ_3 are unknown.

(h) Specialize the calculations in (d) - (g) to the case when $Z \sim$ Bernoulli(θ_4) and compare the information bounds.

4. **Optional bonus problem:** Lehmann and Casella, problem 5.21, page 139.

5. **Optional bonus problem:** Read Note 8.5, Lehmann and Casella, page 145. Explore the identity in the second display in this note and see if it makes sense as written. If not, rewrite the identity in a way that makes sense to you. [Compare with Efron and Johnstone (1990) and/or Bickel, Klaassen, Ritov, and Wellner (1993), pages 420-424.]