Statistics 581, Problem Set 7

Wellner; 11/9/2005

Reading: Chapter 3, Section 2;
Ferguson, ACILST, Chapter 19, pages 126-132, Chapter 20, pages 133-134;
Lehmann and Casella, pages 113-129, and 439- 443.
Due: Wednesday, November 16, 2005.

1. Compute and plot the *score for location*, $-(f'/f)(x)$ when:
   A. $f(x) = \phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, (normal or Gaussian);
   B. $f(x) = \exp(-x)/(1+\exp(-x))^2$, (logistic);
   C. $f(x) = \frac{1}{2}\exp(-|x|)$, (double exponential);
   D. $f = t_k$, the $t-$distribution with k degrees of freedom;
   E. $f(x) = \exp(-x)\exp(-\exp(-x))$, Gumbel or extreme value.

2. Compute $I_f = \int (f'(x)/f(x))^2 f(x)dx$, the information for location, for each of the densities in problem 1.

3. Consider the two parameter location-scale model

$$\mathcal{P} = \{P_\theta : \frac{dP_\theta}{d\lambda} = p_\theta : \theta \in \Theta\}$$

   where $\Theta = \mathbb{R} \times \mathbb{R}^+$,

$$p_\theta(x) = \frac{1}{\theta_2} f\left(\frac{x-\theta_1}{\theta_2}\right),$$

   and the (known) density f has a derivative f' almost everywhere with respect to Lebesgue measure λ.
   (a) Calculate the information matrix $I(\theta)$ for θ.
   (b) For which of the densities in A-E of problem 1 is $I_{12}(\theta)$ not zero?

4. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \le m < k$. Let $\dot{l} = \dot{l}_\theta = (\dot{l}_1, \dot{l}_2)$ be the corresponding partition of the (vector of) scores $\dot{l}$, and, with $\widetilde{l} \equiv I^{-1}(\theta)\dot{l}$, the *efficient influence function* for θ, let $\widetilde{l} = (\widetilde{l}_1, \widetilde{l}_2)$ be the corresponding partition of $\widetilde{l}$. In both cases, $\dot{l}_1$, $\widetilde{l}_1$ are $m-$vectors of functions, and $\dot{l}_2$, $\widetilde{l}_2$ are $k-m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

   where I_{11} is $m \times m$, I_{12} is $m \times (k-m)$, I_{21} is $(k-m) \times m$, I_{22} is $(k-m) \times (k-m)$. Also write

$$I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}.$$

   Verify that:
   A. $I^{11} = I_{11\cdot 2}^{-1}$ where $I_{11\cdot 2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21}$,

$I^{22} = I_{22\cdot 1}^{-1}$ where $I_{22\cdot 1} \equiv I_{22} - I_{21} I_{11}^{-1} I_{12}$,
$I^{12} = -I_{11\cdot 2}^{-1} I_{12} I_{22}^{-1}$,
$I^{21} = -I_{22\cdot 1}^{-1} I_{21} I_{11}^{-1}$.

This amounts to formulas (5) and (6) of section 3.2, page 15.

B. Verify that

$\widetilde{l}_1 = I^{11}\dot{l}_1 + I^{12}\dot{l}_2 = I_{11\cdot 2}^{-1}(\dot{l}_1 - I_{12} I_{22}^{-1}\dot{l}_2)$, and
$\widetilde{l}_2 = I^{21}\dot{l}_1 + I^{22}\dot{l}_2 = I_{22\cdot 1}^{-1}(\dot{l}_2 - I_{21} I_{11}^{-1}\dot{l}_1)$.

The first of these is (7) on page 15, section 3.2.