Statistics 581, Problem Set 4

Wellner; 10/19/2005

Reading: Course Notes, Chapter 2, sections 3-6; Ferguson, ACILST pages 44 - 66.
Due: Wednesday, October 26, 2005.

1. Suppose that $\underline{N}_n \sim$ $\text{Mult}_k(n, \underline{p})$ and $\widehat{p} = \underline{N}_n/n$. Suppose that the true $\underline{p}$ is $\underline{p}_n = \underline{p}_0 + n^{-1/2}\underline{c}$ where $\underline{1}^T\underline{c} = 0$. Use the Cramér - Wold device together with either the Liapunov or the Lindeberg-Feller CLT to show that

$$\underline{Z}_n = \left(\frac{N_{n,1} - np_{0,1}}{\sqrt{np_{0,1}}}, \ldots, \frac{N_{nk} - np_{0,k}}{\sqrt{np_{0,k}}} \right)$$

satisfies $\underline{Z}_n \rightarrow_d \text{diag}(1/\sqrt{p_0})\underline{c} + \underline{Z}$ where $\underline{Z} \sim N_k(0, I - \sqrt{p_0}\sqrt{p_0}^T)$. (It therefore follows, as outlined in class, that the chi-square statistic $Q_n \rightarrow_d \chi^2_{k-1}(\delta)$ with $\delta = \sum_{j=1}^k c_j^2/p_{0,j}$ under the local alternative $\underline{p}_n$.)

2. Ferguson, ACILST, problem 5, page 50.

3. Ferguson, ACILST, problem 4, page 55.

4. Suppose that $X_1, X_2, \ldots$ are i.i.d. positive random variables, and define $\overline{X}_n \equiv n^{-1}\sum_{i=1}^n X_i$, $H_n \equiv 1/(n^{-1}\sum_{i=1}^n (1/X_i))$, and $G_n \equiv \{\prod_{i=1}^n X_i\}^{1/n}$ to be the *arithmetic, harmonic, and geometric* means respectively. We know that $\overline{X}_n \rightarrow_{a.s.} E(X_1) = \mu$ if and only if $E|X_1| < \infty$.
(a) Use the SLLN together with appropriate additional hypotheses to show that $H_n \rightarrow_{a.s.} 1/\{E(1/X_1)\} \equiv h$, and $G_n \rightarrow_{a.s.} \exp(E\{\log X_1\}) \equiv g$.
(c) Use the multivariate CLT and the delta method to find the joint limiting distribution of $\sqrt{n}(\overline{X}_n - \mu, H_n - h, G_n - g)$. You will need to impose or assume additional moment conditions to be able to prove this. Specify these additional assumptions carefully.
(d) Suppose that $X_i \sim \text{Gamma}(r, \lambda)$ with $r > 0$. For what values of r are the hypotheses you imposed in (c) satisfied? Compute the covariance of the limiting distribution in (c) as explicitly as you can in this case.
(e) Use the result in (c) to show that $\sqrt{n}(G_n/\overline{X}_n - g/\mu) \rightarrow_d N(0, V^2)$ and compute V^2 explicitly when $X_i \sim \text{Gamma}(r, \lambda)$ with r satisfying the conditions you found in (d).

5. Suppose that $Y_i = \alpha + \theta'(x_i - \overline{x}) + \epsilon_i$, $i = 1, \ldots, n$, where $\epsilon_i \sim (0, \sigma^2)$ are i.i.d. and the x_i's are known vectors in R^k. Equivalently, $\underline{Y} = X\underline{\beta} + \underline{\epsilon}$ where

$$X^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 - \overline{x} & x_2 - \overline{x} & \cdots & x_n - \overline{x} \end{pmatrix}$$

so that X is an $n \times (k+1)$ matrix. Let $\hat{\underline{\beta}}$ be the least squares estimator of $\underline{\beta} = (\alpha, \theta')'$; i.e. $\hat{\underline{\beta}} = (X^TX)^{-1}X^T\underline{Y}$. Suppose that $n^{-1}(X^TX) \rightarrow D$ where D is positive definite.

(a) What additional condition(s) do you need to impose to prove that

$$\sqrt{n}(\hat{\beta}_n - \beta) \to_d N_{k+1}(0, \text{“}\textit{something}\text{”})\,?$$

(b) Find “something” in part (a).

6. **Optional bonus problem:** Ferguson, ACILST, problem 4, page 49.

7. **Optional bonus problem:** Ferguson, ACILST, problem 3, page 66, and note the particular case $g_j(x_j) = \sqrt{x_j}$, $j = 1, \ldots, k$.